THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY
T. A. A. BROADBENT, M.A.
ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10.

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OF GEOMETRIE
of the moſt aunci-
ent Philoſopher
EVCLIDE
of Megara.

Faithfully (now firſt) tranſlated into the Engliſhe toung, by H. Billingſley, *Citizen of London. Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the beſt Mathematicians, both of tome paſt, and in this our age.*

With a very fruitfull Preface made by M. I. Dee, *ſpecifying the chiefe Mathematicall Sciences, What they are, and whereunto commodious: where, alſo, are diſcloſed certaine new Secrets Mathematicall and Mechanicall, until theſe our daies greatly maſſed.*

Imprinted at London by *Iohn Daye.*

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THE FIRST ENGLISH EUCLID.

THE first complete Euclid in an English translation appeared in 1570. The translator was Sir Henry Billingsley; notes and quotations from all the important commentaries are included in this "monumental work of 928 pages in folio", as well as John Dee's "fruitfull Preface, specifying the chiefe Mathematicall Sciences, what they are, and whereto commodious; where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, vntill these our daies greatly missed".

Billingsley, sometime Scholar of St. John's College, Cambridge, was Sheriff of London in 1584 and Lord Mayor, on a death vacancy, in 1596. Dee, also a Johnian, but some ten years senior to Billingsley, Fellow of John's and then one of the original Fellows of Trinity, studied and taught at various Universities abroad (his lectures in Paris on Euclid are said to have been given in English, a statement somewhat difficult to believe), declined an invitation to lecture on mathematics at Oxford, and made a considerable reputation as astrologer and alchemist at Elizabeth's court. Perhaps because of Dee's notoriety, a good deal of credit for this translation has often been given to him, as for instance by De Morgan, and also to an Augustinian friar, Whytehead, who appears to have been resident with Billingsley for a time. But, according to Rouse Ball, the copy of the Greek text of Theon's Euclid which Billingsley used contains notes, comments and emendations in his own hand which make it clear that whatever may have been contributed by Dee and Whytehead, the main work was done by Billingsley himself.

Two other small points of interest may be mentioned. The title page shows the common contemporary confusion of Euclid the geometer with the philospher Euclid of Megara. In Book XI, in addition to the diagrams, pieces of paper are "pasted at the edges on to the pages of the book so that the pieces can be turned up and made to show the real form of the solid to be represented".

For the photograph, we are indebted to the British Museum Library and Photographic Department.

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GEOMETRICAL DRAWING IN THE HIGHER SCHOOL CERTIFICATE COURSE IN SCHOOLS.*

The Introduction of Geometrical Drawing into the Higher School Certificate Course.

Dr. W. G. Bickley : In the first place some explanation may be looked for as to why one so apparently unsuitable should be entrusted with the task of opening this discussion. It is years since I was associated with school teaching; at present drawing is difficult, and proper instruction of pupils, and criticism of their work, would be almost impossible. Admittedly I was rash enough to try to help the programme secretary by proposing this subject for discussion, and then the Council would not let me escape the punishment for my temerity.

Seriously, however, I do believe most strongly in what I am here to advocate, and would hope that this meeting might send to all the H.S.C. examining bodies in the country a resolution strongly and unanimously urging them to include geometrical drawing in the H.S.C. mathematics syllabuses—but let me begin at the beginning.

I was brought up in a small town where a rather specialised branch of engineering is carried on. Thanks to the encouragement of an enlightened vicar, excellent evening classes for apprentices and others had been established some hundred years ago and were well patronised. As a consequence I had, from father and uncles, a legacy of books covering a wide range of subject. Among them were several upon " Practical Plane and Solid Geometry ", a subject upon which draughtsmen and other craftsmen were nurtured in the pre-Hall and Stevens' *School Geometry* days. While still at the elementary school I had begged as a birthday present a drawing-board upon which to use father's instruments in working out the fascinating problems in these books. When I did get to the secondary school, the knowledge I had gained in this way enabled me to see the sense of " Hall and Stevens ". Indeed, the replacement of the mystery of dogma by the enlightenment of reason was an intellectual thrill. Nowadays, of course, everybody pays lip service to a preliminary practical and experimental stage in geometry, but in my day the textbooks and M.A. reports had not been written. And I am still prepared to assert that those who followed the old Science and Art Department courses on Practical Plane and Solid Geometry did in fact know, and could *use*, more real geometry than a large proportion of our S.C. and H.S.C. candidates.

I do not remember much encouragement of geometrical drawing at my secondary school. But when I went to Reading, the late Herbert Knapman's course on Geometry included the working of all the exercises in the early chapters of Filon's *Projective Geometry*. Neither the text nor the exercises were easy; either without the other would have been virtually impossible. The exercises were to be carefully worked on the drawing-board—a sweat, but Knapman was inexorable, and, as we soon came to realise, we were rewarded by the grip of the subject that we thereby obtained.

My teaching has been largely associated with engineering students. Although —or because—drawing forms an integral part of their training, it is noticeable how favourably they always react to exercises solved by drawing, from graphic statics to differential equations. I have previously † given an account of possibilities in these directions. But what I here wish to emphasise is that

* A discussion at the Annual Meeting of the Mathematical Association, 24th April, 1946.

† *Gazette*, XXV (May, 1941), p. 87.

these students form a cross-section of the saner and more balanced majority of the school population, and that their likes and reactions may be regarded as a fair sample.

Now for the fact which led to my suggesting this discussion. As most of you know, entrance to the University *via* H.S.C. and exemption from an Intermediate Examination is becoming increasingly common. You also know that Intermediate Engineering has included the subject of " Machine Drawing ", exemption from which through H.S.C. is difficult or impossible. The problem was considered by the Faculty of Engineering of the University of London, and the decision, after discussion with the Faculty of Science, is that there shall be a common Intermediate in Science and Engineering. The engineers give up the subject of " Machine Drawing ", but the scientists welcomed the suggestion that the revised syllabuses should contain geometrical drawing as part of the mathematics. This decision was the trigger, but the force of my contentions derives, I believe, from considerations more general and more fundamental.

First, let us consider the evolution of man's mental activity. In the early forms of life, sensation leads directly to action. Later, brain develops, and begins to inhibit or direct action; but it is still action which is ultimate. Abstract and detached thought comes at a very late stage in evolution, and still occupies only a very small portion of the mental life of the most intellectual among us. It would therefore seem incontrovertible that in all children, most adolescents, and many adults, it is natural that thought should lead to action, and almost unnatural that thought should be an end in itself. Our subject suffers more than most school subjects from the difficulty that it requires much thought, but offers relatively little opportunity for thought to culminate in action. It is therefore essential that we teachers of mathematics should take advantage of every opportunity—and even go out of our way to create opportunities—of *doing* as distinct from merely thinking. Clearly geometrical drawing offers one of the more obvious and more fruitful ways in which we can do this.

My second point is closely connected with my first. Muscular and sensory discrimination are just as necessary to a full life as is intellectual. Mathematicians and philosophers with active heads and useless hands are objects of pity—when not objects of derision. Our pupils will be more, and not less, alive if they can use their hands and eyes as well as their heads. Geometrical drawing provides discipline necessary for the acquisition of skill, as well as opportunities for that satisfaction which comes from its exercise. Sheer virtuosity may here be encouraged.

My next point is that action stimulates the intellect—the desire to do is one of the strongest incentives to thought. A problem actually to be worked on the drawing-board will much more readily cause the pupil to marshal and use his or her geometrical knowledge than will the corresponding " rider " where the construction is only to be discovered and described. For most pupils, and in the long run, the time actually devoted to the drawing will be more than saved by the better grip of the subject obtained and by the enhancement of interest. The reality of a tangible drawing neatly and carefully executed will drive home many points which would otherwise be nebulous or missed. Even if abstract thought is our final goal, surely the provision of a greater variety of experience from which to abstract is a means of increasing our chance of success.

But already some of you may be asking what sorts of things we may set our sixth forms to do. First of all, there is much of interest and value that can be done with harmonic ranges, pole and polar, coaxal circles, and cognate topics. The harmonic properties of the complete quadrilateral come to life

when they have been used, say, to construct the polar of a given point with respect to a given circle. Then there are all the various methods of constructing conics to satisfy prescribed conditions. The two-circle method for an ellipse, for example, is a direct use of the parametric equation $x=a\cos\theta$, $y=b\sin\theta$. Other methods depend on Pascal's or Brianchon's theorem. A simple case of a conic drawn as an envelope of tangents is given in my first slide (Fig. 1). It would be good to draw, in pencil, at least three times as

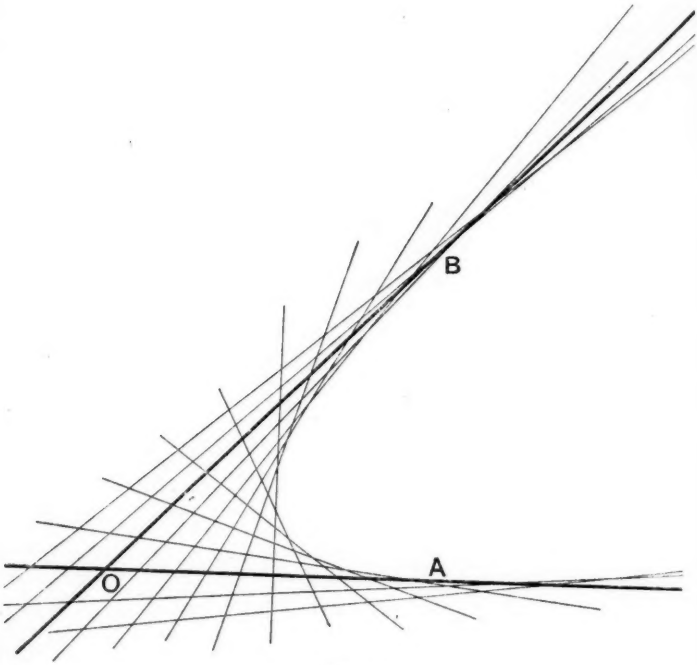

FIG. 1.

many tangents, but such a figure would not reproduce well. Aesthetically such a figure is most pleasing, and gives a challenge to virtuosity to which most pupils would rise, and where self-criticism is inevitable.

Then there are many aspects of descriptive geometry. I leave details to Dr. Weikersheimer. What I wish to emphasise is the variety of methods of representing three dimensions in two, a variety of device and convention, each with merits and deficiencies, each a language and a symbolism which, when learned, enormously enhances one's ability to think in three dimensions. I wonder how many will immediately read the message intended to be conveyed by my second slide (Fig. 2). It is, of course, plan and elevations of a solid letter **H**. Does the next slide (Fig. 3) convey this more immediately? If so, what demerits has isometric projection compared with plan and elevation? Then (Fig. 4) another variation of the same theme—this time what

seems to me the very satisfying idiom of geometrical perspective. I have, by the way, deliberately omitted all construction lines.

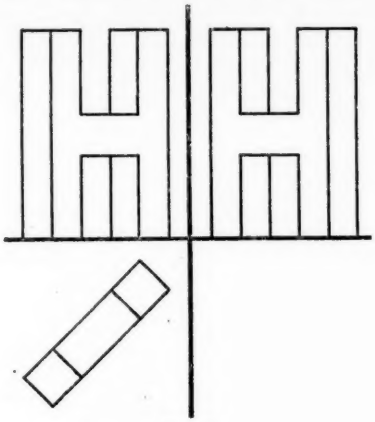

FIG. 2.

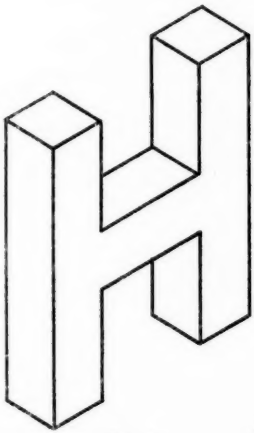

FIG. 3.

Having learned the various techniques, the pupils should *use* them to solve the various problems. I content myself with one simple case, determination of the form of the shadow of a rectangular board $ABED$ by a light source at L (Fig. 5). This kind of problem has, in my opinion, all the educational virtues. As additional exercises, which I leave to my hearers, I suggest working the same problem in isometric projection, and in geometrical perspective. These few illustrations should suffice to indicate the possibilities of the subject.

I could also remark that this is an age of blue-prints, and emphasise the desirability that our pupils should be able to read them, as well as the fact that the desire of our pupils to do so can provide a strong incentive to sustain interest in this work.

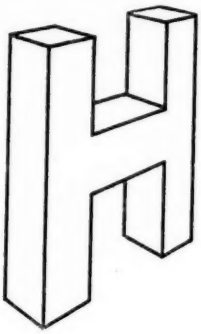

FIG. 4.

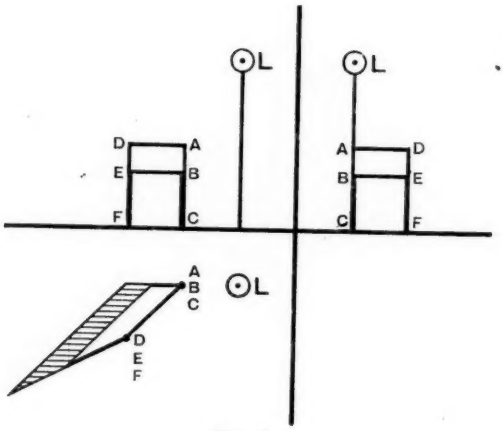

FIG. 5.

But I do not rest my case upon any utilitarian grounds. I commend to you the subject of Geometrical Drawing in the H.S.C. course for three main reasons:

(i) that thought should lead to action;

(ii) that the proper place for geometry is on the drawing-board;

(iii) that any right-minded teacher or pupil, of either sex, should derive much pleasure from a course such as I have outlined.

On Descriptive Geometry and its Educational Value.

Dr. S. Weikersheimer (King Edward's School, Birmingham) : In his basic work, *Géometrie Descriptive*, published 1801, Gaspard Monge outlined the task of this branch of mathematical science as the representation of three-dimensional objects in a plane and the deduction of properties of the objects from their projections. In his lectures he solved a number of problems of solid geometry in this graphical way by the methods of plane geometry, and dealt with applications of descriptive geometry, *e.g.* the construction of shadows and the perspective. E. L. Ince states in his *Principles of Descriptive Geometry* (1933) : a drawing or a photograph will show the object from a definite point of view, but it would be quite inadequate to make the requirements of an architect or engineer clear to those responsible for carrying out his plans.

Before G. Monge was appointed professor of mathematics and physics in the military school at Mézières in 1768, he was employed there as a draughtsman. He had to design fortifications. The solution of such problems involved a great number of laborious computations, so that the ingenious young mathematician discovered a new method : the geometrical solution with plan view and front elevation. It took some time until the simplicity, the utility, and the great value of his principles were recognised and appreciated. Then, his method was considered a military secret, and Monge was not allowed to disclose it, except to the students of that military school (see *Descriptive Geometry* by H. W. Miller, Chapman & Hall, London, 1941). This short story shows that a utilitarian purpose was the motive behind the origin and creation of descriptive geometry. And even now, after one and a half centuries, when the ideas of descriptive geometry have developed, and were adapted to the needs of modern industry, Monge's principles still form the basis of all improvements and enlargements. Therefore it is no wonder that, nearly all over the globe, descriptive geometry was introduced as a compulsory subject into the curriculum of all those students whose future professional work, in its design or its execution, is based on it. Engineering drawing is applied descriptive geometry. But it is not only the utilitarian purpose for which we should teach this subject in the Higher Forms of secondary schools, it is also for its promotion of logical training, for its special contribution to the mathematical education of the youth, in particular the development and strengthening of the power of space intuition, and the knowledge of relations of points, lines, planes, and surfaces in space. Last, not least, this subject—like none other —is inducing the student to tidy and clean, systematic and accurate work, factors which are all of outstanding educational importance. The boys to be mathematically educated should be enabled to read a drawing, to visualise the object, to reproduce a mental picture so that everyone who is skilled in this science can interpret it. Beyond that aim, the pupils should be able to solve problems of solid geometry graphically ; this requires the making of precision drawings. It is nearly always possible to check the correctness and accuracy of drawings in one way or another. More frequently than in any other geometrical subject, we can solve a more complicated space problem by reducing it in a simple way to a problem previously dealt with, or to a special case which can be solved more easily. I shall illustrate this by solutions of problems which don't represent a systematic treatment of descriptive geometry—even not a very concise one—which are selected to demonstrate the principles, methods and devices of descriptive geometry, considering in first line space problems.

From the pedagogical standpoint it is extremely important to revise and enumerate the " reminders " to be used frequently, *e.g.* parallel lines, parallel planes, parallel line and plane ; perpendicular line and plane, perpendicular

planes ; representation of a plane ; angle of inclination of a line to a plane, angle between two planes. It is advisable to discuss (or to revise) a method of slope parallel projection so that the pupils can draw proper diagrams of bodies, if necessary. These figures are very often a great help to the imagination. Furthermore, a system of two (or three) perpendicular planes, made of wood or cardboard, is sometimes very useful.

There are different kinds of projections. According to the technological method—as it was used by Monge, and as it is adopted by engineers and architects, and by teachers in the classroom—the figures in space are orthogonally projected on two or three planes which are perpendicular to one another. The coordinate planes are called horizontal or H plane, vertical or V plane, and profile or P plane ; the projections (representations or descriptions) are the plan view, the front elevation, and the profile (side) view respectively. The line of intersection of the H and V planes is named the ground line GL (or folding line), that of the V and P plane the profile ground line G_1L_1. In order to get the three projections in one plane, we revolve the H plane about the GL into coincidence with the V plane, and the P plane is to be revolved about the G_1L_1 until it coincides with the V plane. The H and V projections of a point are therefore on a perpendicular to the GL, and the V and P projections on a perpendicular to the G_1L_1. In the biorthogonal projection, there are nine different positions of a point possible ; they constitute the " alphabet of a point ". Point A is in the first quadrant, etc., point E is a point in the positive H plane, point H in the negative V plane. Often it is useful to draw first the figure in parallel projection (or a perspective figure), and then the figure as it is done in descriptive geometry. In a similar way we can state an " alphabet of a straight line ", *e.g.* perpendicular to the H, or parallel to the H plane and inclined to the V plane, etc.

It will be useful to deal with certain problems as fundamental problems to which the students may refer. I would suggest : (1) To find the traces of a straight line, (2) the traces of a plane. Converses (as exercises) would be: (*a*) the traces of a plane are given, and one projection of a point or a straight line, find the other projection so that the point or the straight line is in the plane ; (*b*) find the projections of an n-sided plane figure ($n > 3$).

(3) To find the piercing point of a straight line and a plane. In most problems where a plane is given you should assume that the plane is determined either by its traces or by two intersecting straight lines (or by three points). (4) Drop the perpendicular x to a plane from a point P (Fig. 2).

I shall briefly explain how to solve this problem. Let x be the required perpendicular. x_1 is its H projection, l a projecting line. Since l is perpendicular to the H plane, the plane (xl) is perpendicular to the H plane (theorem); since x is perpendicular to the plane (s_1t_2), the plane (xl) is perpendicular to the plane (s_1t_2) ; therefore the plane (xl) is perpendicular to s_1 because the plane (xl) is perpendicular to two intersecting planes, therefore it is perpendicular to the line of intersection ; thus s_1 is perpendicular to any straight line in the plane (xl), so s_1 is perpendicular to x_1. Similarly, one can prove that t_2 is perpendicular to x_2. Therefore the theorem : If a straight line is perpendicular to a plane, the projections of the straight line are perpendicular to the like traces of the plane. The solution of the fundamental problem (4) is now easy. You have only to draw the perpendiculars to s_1 and t_2 through P_1 and P_2 respectively. If the plane (s_1t_2) is not given by its traces, the problem can be reduced to it by finding them, or it will be sufficient to have parallel lines to the traces.

You will realise that in the solving of this comparatively simple problem of descriptive geometry the mental effort involved is at least equal, and the imagination required, certainly greater than those of similar problems in plane geometry.

The pupils have to apply theorems of solid geometry, which will become quite familiar to them if they were not before, and to correlate them as they used to do in plane geometry. So, such problems of descriptive geometry

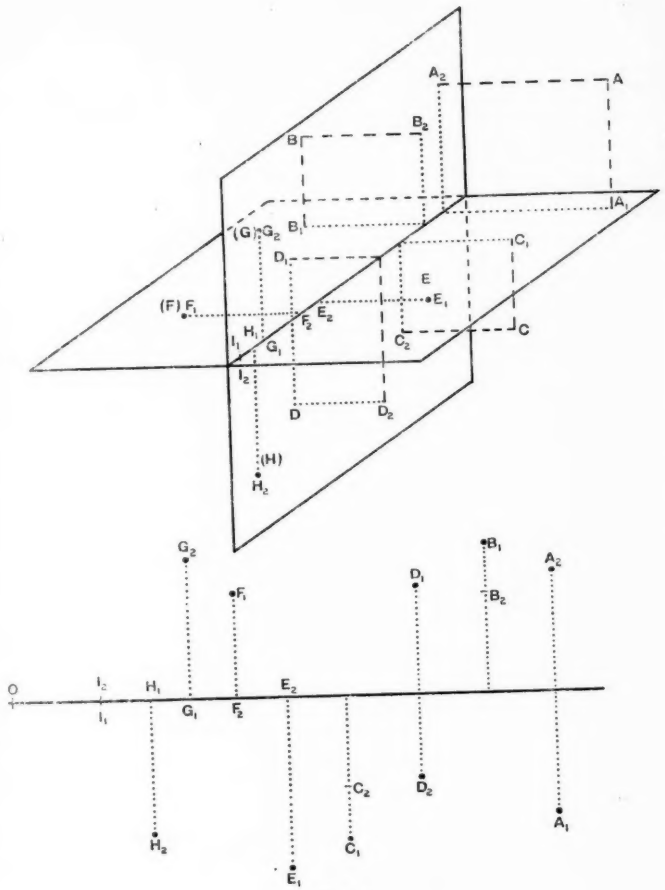

FIG. 1.

share in the logical training of our youth in the same way as any other mathematical branch, and they will involve the key to the solution of problems of greater practical value.

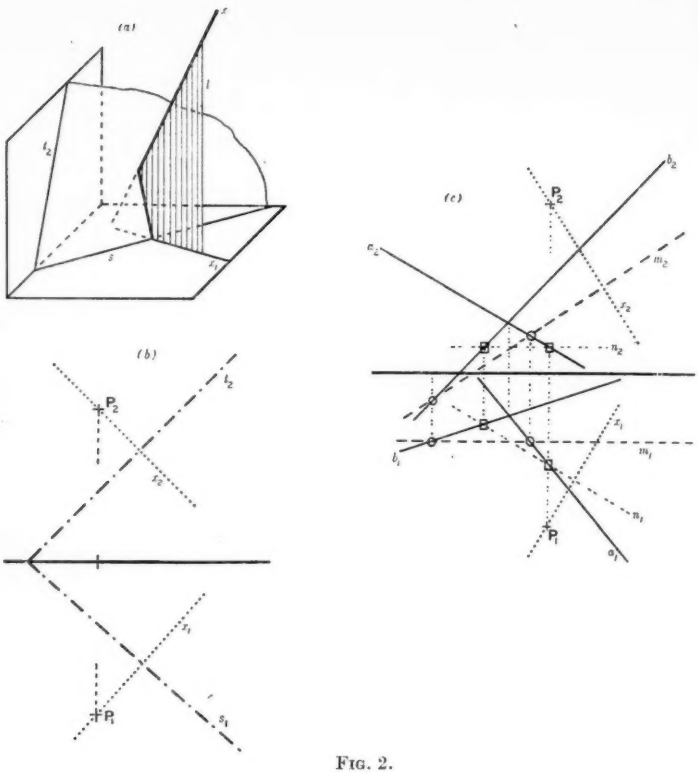

FIG. 2.

Other fundamental problems should be (a converse to 4):

(5) Pass the plane through the point P, perpendicular to the straight line l.

(6) To find the line of intersection of two planes. The planes can be given by different elements, and the methods of solving can be different. A practical application of this problem is shown in the intersection of a triangle and a plane quadrilateral under the consideration of visibility and invisibility.

A very instructive, more abstract, example is the fundamental problem (7): To pass through the point A a straight line x which cuts two skew straight lines a and b (Fig. 4).

Analysis to the solution: Since x passes through A and cuts a and b, x must be in the planes (Aa) and (Ab), therefore x must be the line of intersection of the planes (Aa) and (Ab); or: x is in the plane (Aa), and x cuts b, therefore x must pass through the point Y of intersection of b and the plane (Aa). In the construction of the second method, pass the auxiliary line m through A, which cuts a; the plane (Aa) is the same as the plane (am). b intersects (am) at Y.

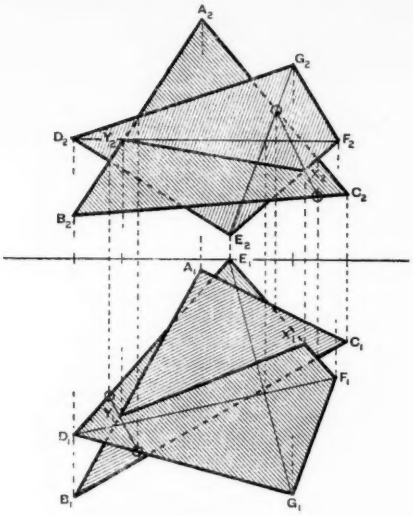
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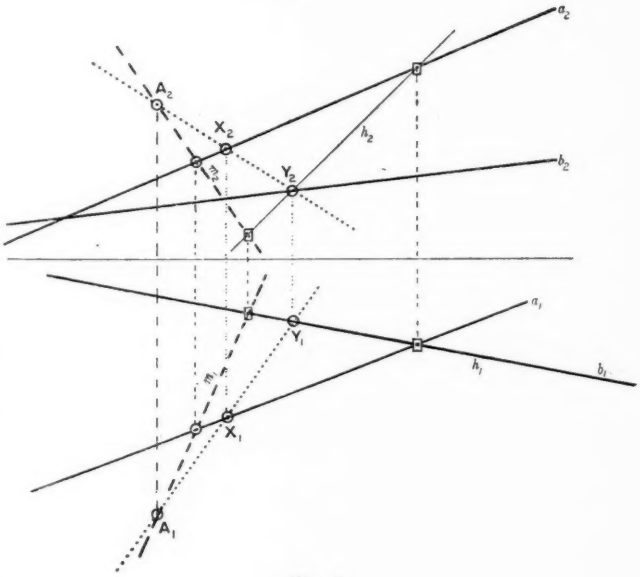
FIG. 3.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A1

FIG. 4.

(8) Find the straight line x cutting the two skew straight lines a and b at right angles, or : to find the shortest distance between the two skew straight lines a and b.

There are also different methods of solution possible. I want to sketch the following one by reducing this problem to the previous one. (On page 14, Fig. 7, I shall solve the same problem once again, but the solution will be based on quite a different principle.)

Analysis : Since x is to cut a and b at right angles, x must lie in planes perpendicular to a and to b. There is an infinite number of such planes, their lines of intersection (i) are all parallel to one another. In this way the problem is reduced to No. 7, the point A of (7) is replaced by a point at the

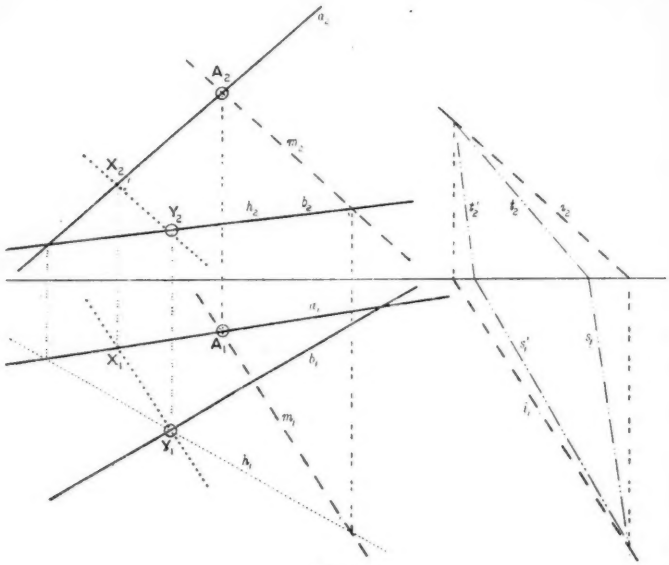


FIG. 5.

infinity, *i.e.* by the direction of i. In the construction we draw first the traces of the planes perpendicular to the straight lines a and b. These two planes (s_1t_2) and ($s_1't_2'$) cut each other in i. The auxiliary line m is drawn parallel to i, therefore the plane (am) is parallel to the straight line i. The straight line b pierces the plane (am) at Y. Draw $x \equiv XY$ parallel to m.

Some more fundamental problems can be solved by applying the principle of revolution, *e.g.* :

(9) to find the true length of the distance AB, and to find the angles of inclination of the straight line AB to the H and V plane ;

(10) to find the true shape of the triangle ABC.

In (9) the projecting plane ABB_1A_1 is revolved about A_1B_1 until the plane coincides with the H plane, or ABB_2A_2 is revolved about A_2B_2 into the V plane. We call the distances obtained A_3B_3 or A_4B_4 respectively. A_3B_3 must be equal to A_4B_4, equal to AB, the true length. If we produce

A_3B_3 to meet A_1B_1 we obtain the angle of inclination of AB to the H plane. The true length of the distance can also be found by revolving the distance about A or B until AB is parallel to the H or V plane.

The solution of (10) involves the revolution of the plane ABC about its trace s_1 into the H plane ; for this purpose you must find the angle of inclination of the plane to the H plane. Notice the relation of affinity between the projection $A_1B_1C_1$ and the true shape $A_3B_3C_3$; the points of intersection of A_1B_1 and A_3B_3, A_1C_1 and A_3C_3, B_1C_1 and B_3C_3 are collinear (Desargues). The principle of revolution is generally to be applied to the solution of problems where the true shapes of figures, the true lengths of distances, or the true sizes of angles are required.

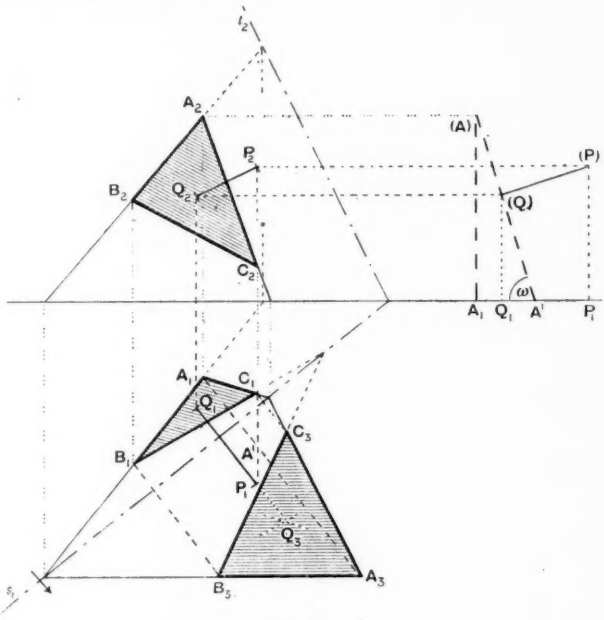


FIG. 6.

The figure No. 6 also shows the construction of the circumcentre Q of the triangle ABC. We first construct Q_3, and by means of the triangle $(Q)Q_1A'$ we get the H and V projections Q_1 and Q_2.

The triangle $(A)A_1A'$ represents a plane perpendicular to the H plane, and perpendicular to s_1, therefore it is perpendicular to the plane (s_1t_2). A perpendicular to the plane of the triangle ABC at Q is parallel to the plane $(A)A_1A'$. Therefore $(Q)(P)$, the perpendicular to $(A)A'$ at (Q), is the true length of the perpendicular to the plane ABC, erected at Q. The projections of the point P, P_1 and P_2, can easily be found by means of the auxiliary figure $(Q)(P)P_1$. In this way, the 11th fundamental problem can be solved, viz. : Find the perpendicular distance of the point P from the plane ABC.

The introduction of a third projection plane as an auxiliary plane is quite frequently of great help for the solution of problems in descriptive geometry. We can consider this method as a principle ; *e.g.* the 8th fundamental problem (to find the shortest distance between two skew straight lines) can be solved in the following way by applying this principle to it. Assuming the straight line a is perpendicular to the H plane, the perpendicular from a_1 to b_1 is the required distance. X_2Y_2 is parallel to the G.L. The general case (where a is not perpendicular to the H plane) can be reduced to the special case—which method can also be considered a principle in descriptive geometry—by introducing an auxiliary plane perpendicular to the straight line (a) so that the projection of (a) into this auxiliary plane is a point (a_3). Then we have to project b on that plane ; the perpendicular from a_3 to b_3 is the true length of the shortest distance between the skew lines a and b. By " projecting back " we obtain the H and V projections of X and Y.

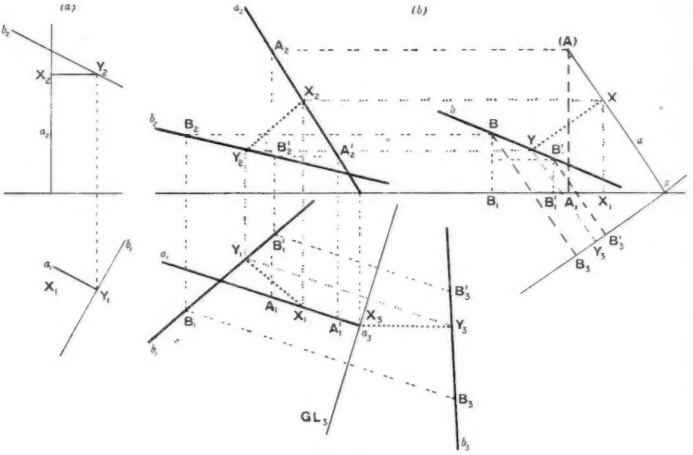


FIG. 7.

Further fundamental problems are :

(12) to find the angle of two intersecting straight lines, which problem is generally solved by revolution about a trace of the plane of the two intersecting straight lines ;

(13) to find the angle of two planes. This problem can be solved by the introduction of an auxiliary plane perpendicular to both given planes, therefore perpendicular to the line of intersection.

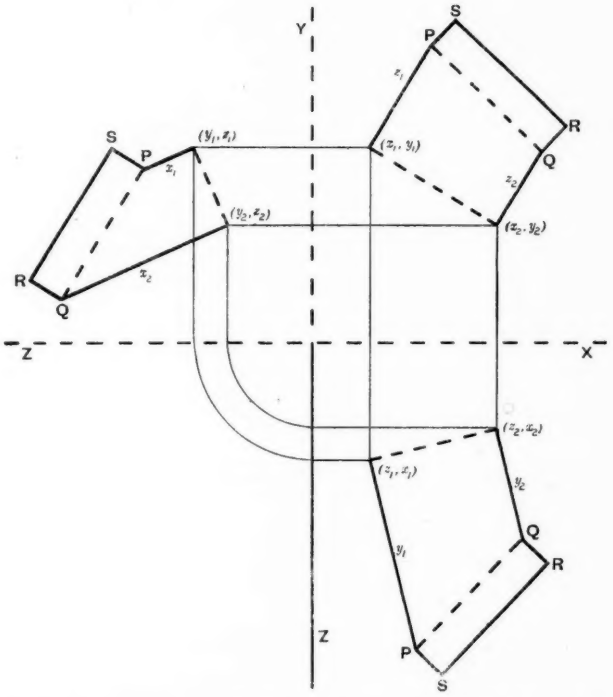
Of course, the number of fundamental problems can be increased, or other important problems must be dealt with as exercises ; in particular some problems on loci should be solved.

The problems on points, straight lines, and planes are usually followed by the representation of plane and curved surfaced bodies (prism, pyramid, cylinder, cone, and sphere), and the intersection of them by straight lines, planes, and by one another. Further problems are the construction of tangent planes, development of bodies, shadows, and finally the perspective.

The **President (Prof. S. Chapman)** felt sure all agreed that it had been interesting to hear the account of how descriptive geometry figured in the English school and what had been done in Bavaria. It was a pity a Frenchman had not come to describe what was given in the way of descriptive geometry in France.

Mrs. E. M. Williams (Leicester Training College) could not help recalling, when looking at Dr. Bickley's first diagram of the envelope, the occasions on which quite mature young men and young women had made that envelope, not on a drawing-board or with a pencil and rule, but with brightly-coloured threads and the use of a needle. It was, perhaps, fitting that there should be a feminine reminder that straight lines could be sometimes picturesquely drawn with needle and thread.

Mr. E. Barton (Radley) said he once did a certain amount of descriptive geometry, not with post-certificate forms, but with a form in the year before the school certificate year. He was not sure whether he hit on the device illustrated below or whether someone told him of it, but at any rate it was useful as a means of giving confidence in the results of drawing plans and elevations.



Cut along heavy full lines, fold along dotted lines, clipping third quadrant behind fourth and the three flaps *PQRS* together.

Mr. B. J. Banner (Chichester) reminded members that Dr. Bickley had said this was a blue-print age, and they would, the speaker thought, agree readily that a blue-print was merely a means to an end, the end always being the same, some solid construction, some piece of engineering work. With that in mind members should be on their guard in regard to geometrical drawing, lest they made use of plane drawing as an end in itself. It was fascinating to boys to be able on a flat piece of drawing-paper to obtain the projections in different planes, to show the appearance of some solid figure, for example, from different aspects, and that was useful; but it should in all cases be followed by an attempt to produce the actual solid figure or the collection of points concerned.

To many present it would be a commonplace suggestion that more use might well be made of the Euclidean solid figures as an introduction to that type of geometrical drawing. To take possibly the simplest example, it was an easy matter to draw the development of the cube and to persuade even young pupils to then proceed to fold up the cube and produce the block, and one might well follow with the usual five regular solids which so many teachers would agree pupils found extremely attractive. In all those cases the working out of the plane development was interesting in itself. It provided an extraordinarily valuable exercise in accurate drawing, because, obviously, if a drawing was not accurate neither was the model; and, moreover, it was altogether justified by the Euclidean textbooks, because it formed the basis of the final two or three books. There was hardly any limit to that type of work in the case of pupils who showed a special interest in it. One might pass on to the Archimedean solids, to semi-regular figures and the host of others derived from them. The point was that in all cases in that type of work one had the thought passing into action in several ways. There was the actual geometrical drawing and the developments which had been discussed by Dr. Weikersheimer, and there was, finally, the satisfaction, to younger pupils especially, of seeing the completed solid model.

Mr. L. Ehrenberg (St. Bees) had not taught in France, but he had lived in Czechoslovakia, where the school system was similar to that outlined by Dr. Weikersheimer as pertaining to Bavaria. In Czechoslovakia, descriptive geometry was done on a lower plane and by younger pupils; it was taken in the third and fourth years, mainly of the secondary school; that was to say, by boys of 12, 13 and 14. Shortly after the pupils passed through that stage the speaker had come to England, and soon after took school certificate, when he was surprised and a little shocked to find what a little part geometrical drawing had in this country. So many examples started with "Draw a rough diagram", and so many pupils took that literally!

Geometrical drawing was a very valuable exercise, not only in drawing but in geometry. The drawing of solids might consist of first of all the drawing of nets of regular solids, particularly of the cube, the tetrahedron, and then simple plan and elevation of solids. In plane geometry, most important of all was the accurate drawing and conversion of an irregular polygon into the square of equal area. That necessarily must be done accurately, because the two areas must be the same. It was done theoretically, at least in English schools, but the boys on the whole did not appreciate the importance of the exercise because they were not made to do it properly, as it could only be done, on a drawing-board; it could not be done properly in an exercise book.

Also for simple teaching of accuracy and facility of using, for instance, the Indian ink compass, which was not quite so easy for boys of twelve, it was useful to start the course with simple pattern drawing, such things—snakes and so on—as involved circles moving in and out, in which pupils took a great delight.

Unfortunately, the syllabus in this country was pretty well filled up. In the lower forms a course of drawing of that description could not be introduced without overcrowding the curriculum, unless some of the propositions were cut out.

Mr. A. W. Siddons (Harrow) spoke as one who had not taught descriptive geometry for some forty years past. The books from which he had learned had used P and Q for the actual points ; p and q for the projections in the horizontal plane ; p', q' for the projections in the first vertical plane ; p'', q'' for the projections in the second vertical plane ; and if one went on to a third plane, p''', q'''. Had that convention disappeared, or did it still hold?

Mr. A. Turner (Regent St. Polytechnic) replied that the intersection of a vertical plane with a horizontal plane produces four pairs of the reference planes used in geometry. These are numbered 1, 2, 3 and 4, numbers 1 and 2 being in front of the vertical plane and number 1 being above the horizontal plane. The first pair are used in Great Britain, and hence the elevation of any point in this pair of planes is above the XY line, and the plan of the point is below the XY line. A line inclined to both the vertical and horizontal planes might produce, however, when extended, a point in either of the pairs of reference planes numbers 2 and 3, in which case it is necessary to distinguish the elevation of a point from its plan. This is done by attaching a dash at the top of the letter denoting the elevation of the point, for example, a'.

A second elevation of the same point is sometimes distinguished from the first elevation by a double dash, such as a'' ; but perhaps a better notation is a_1' (first elevation), a_2' (second elevation), a_3' (third elevation), etc. Successive plans would be a_1, a_2, a_3, a_4, etc.

In the United States the third pair of planes is used so that the elevation of a point comes below the XY line and the plan above.

Mr. A. J. Hatley (Imperial College) spoke of geometrical drawing and descriptive geometry from the examination point of view. Some years ago he had been an examiner in connection with the Mathematical Drawing examination of the Royal Drawing Society. There were six divisions of drawing for pupils from the ages of twelve to seventeen. Part of the subject was what could strictly be called descriptive geometry. The students, in a given time, had to draw accurately, but there was another section of their work which was freehand. That was by no means rough, but the previous experience of the proper drawing was helpful in the latter case. As a result of the drawing he saw and the work done, the speaker said he was able to testify to the value of that method to the pupil concerned. One of the difficulties was that in the senior divisions many sixteen- and seventeen-year-old pupils could not spend the time for this drawing owing to the claims of the Higher School Certificate work. But they had done very well, and the pupils had, to the speaker's knowledge, greatly enjoyed the work.

Incidentally, the last speaker had touched on one of the problems that had had to be faced, because candidates came not only from English schools, but from Vancouver (British Columbia), Jamaica and other places. The Vancouver pupils adopted the American conventions in regard to plan and elevation notation generally, so that it was necessary to be careful to see that one got what was asked for.

One of the factors valuable in that examination was that the pupils had to do definite things in a short time. Just as in other subjects there was a choice, say seven questions out of ten, so the pupils had a choice in the definite drawing and in the freehand drawing, and it was valuable to see what a pupil could do in a limited time. As was to be expected, most of the candidates came from boys' schools. They certainly did value the training

and, even if they did not go on to engineering as a career and just went on for the sake of the thought and hand-training, it proved distinctly advantageous. In conclusion, Mr. Hatley hoped that, as Dr. Bickley had said, this cultural value of the subject was a point which would be stressed in relation to future work.

Mr. H. E. Piggott (Canford) reminded members that Dr. Bickley had expressed the view that descriptive geometry should be introduced into School Certificate or the Higher School Certificate papers. There is one public examination in which questions on descriptive geometry are set in the mathematics paper, the special entry for the Royal Navy. Many present had experience of training boys both for the H.S.C. and for the Navy examination. The speaker's experience had been that boys who had been through the descriptive geometry course for the naval examination acquired a sense of passing thought into action and, above all, a space sense which the other boys did not get. Was that the opinion of others who had taken the same course?

Mr. J. Logie (Peter Hills School) spoke with great diffidence when he said that at this moment in London there were many thousands of children of both sexes poring over plans and elevations of various types with the practical object in view of making aeroplanes, ships, household articles of all sorts, and so on, in accordance with the instructions accompanying the plans. Although his experience had mostly been with younger children, the speaker felt sure that human psychology was the same at all ages, and that the best way in which to help those children was to start by helping them to understand those plans which they were studying for themselves, so that they were able to use their ability to read the diagrams they had obtained in order to make the things in which they were interested. Later on they would find a pleasure in drawing for themselves diagrams of things they were going to make. The best way from the pedagogic viewpoint was to start with the things that interested the children. In the end they would proceed to descriptive geometry whether they wanted to or not. It seemed best to leave the plane, the straight line and the point and all those weird and uninteresting abstractions and start from the complicated reality that the child first saw, and go into the pure analysis later—at about the stage when the boy or girl began to learn certain applications of the projection, of the circle, plane, and so on.

Miss H. M. Cook (Edge Hill Training College, Ormskirk) said she had had a small amount of experience of teaching geometrical drawing with some women students training to be teachers. About two years ago there was an effort to alter the Mathematics Syllabus for the ordinary course in the North-Western Region, and the request came that something more practical should be included. The qualification of students taking the ordinary course was that they had passed the School Certificate. Some had done H.S.C., some had dropped the mathematics after School Certificate and had done none for a couple of years before they came to the speaker; often she had found it necessary to go back over a good deal of the School Certificate work. The criticism those concerned were trying to meet at that time was that the syllabus of mathematics was very dull for any who had done any H.S.C. work, and that it had practically nothing new in it for the others. There was an endeavour to include something which was mathematical, but which was not likely to have been tackled by either group of students. So those concerned put in what they called "Topics", such as easy astronomy, elementary statistics, etc. The speaker had twice done a certain amount of what she termed "drawing-board mathematics", as a Stage A for higher mathematics. The pupils were set to drawing ellipses, hyperbolae, parabolae, and so on before they had done any coordinate geometry, and then doing some

sections on cylinders at a particular angle, and plans and elevations, and a little using Pascal's theorem on finding an ellipse that went through five points, and things of that description. There had only been one examination on that work, but it would be examined on again during the coming summer. That was an experiment being tried out.

Mr. S. Inman (Isleworth County School) thought it could be said that the amount of descriptive geometry taught in grammar schools in England was extremely small, and, judging by continental schools, there seemed here to be omission to teach what was considered on the continent to be essential. Recently all the university examining bodies had met to frame an alternative syllabus which incorporated some elements of descriptive geometry. No doubt in the near future that syllabus would become compulsory for all schools, so that the question of the teaching of descriptive geometry was one which would have to be faced. How was it to be taught?

During that meeting there had been mention of two different ages at which the subject was commenced. In Bavaria the age was 16 plus, and in Czechoslovakia it was started about 13 plus. The point of approach seemed to be most important. Should one take the point of approach as outlined by Dr. Weikersheimer? It seemed that the method there was similar to a string of Euclidean propositions, and if tackled in that way the speaker doubted whether the subject would be very suitable for the type of pupil those present had in mind. He suggested that instead of dealing theoretically with the point, straight line and plane, it would be better to start with a simple solid such as a cuboid and merely draw its plan and elevation. It would need little imagination to draw a view perpendicular to a diagonal plane, including a diagonal of the cuboid. One might easily deal with such a problem as the inclination of this line and others to the horizontal plane. What the speaker felt was the need of discussion on the method of approach, and perhaps the problem he had mentioned—and others of a similar kind—was a line that might be followed.

Dr. W. G. Bickley, in replying, felt that it was somewhat difficult to summarise the discussion; he would, therefore, just comment upon one or two of its reactions upon himself.

He hoped he had made it clear from the very beginning that he was entirely in sympathy with anything which would widen rather than narrow the real approach; that was the only way in which mathematics could successfully be taught. He welcomed Mrs. Williams' suggestion that the work might occasionally be done in coloured threads, and also the suggestion by a later speaker that it would be well to start with the things children wanted to do and show them how to do them perhaps a little better.

In proposing that subject for discussion he certainly had in mind something very much wider than descriptive geometry. That was one, and an important item of the things he wished to see done, but there was a great deal of more accurate drawing in problems of plane geometry, harmonic ranges, construction of conics, and so on, which was valuable as mathematical training, and not only as a piece of manual dexterity. The isometric projection, and in particular the geometrical perspective were as much to be encouraged as plans and elevations. Most pupils responded to that sort of work. It was fortunate for teachers that they should, and it would be well to canalise and harness all such responses of children.

At the beginning, and for reasons which Dr. Bickley thought he had made clear, his main argument was directed to the H.S.C., but he had been perfectly well aware that if the work was done at the H.S.C. level it would not be possible to escape doing it at the ordinary School Certificate level. Having taught boys and having marked the work of examination candidates, he felt

the standard of actual drawing to be abysmally low, much lower than it ought to be, so something should be done with regard to getting decent, accurate, neat drawings from most pupils.

As the speaker saw it, the most important part of the whole matter was as follows. He believed that in the end what they were out for as teachers of mathematics, and as teachers generally, was to get as far as they could along that road of evolution which had led to abstract thought and the use of man's reason. But if they started with the abstraction, if they did not give their pupils anything from which to abstract, they would not succeed. He was certainly aiming, as he hoped all present were too, at abstract thought in the end; but the greater variety of real practical tangible experience they could give their pupils to abstract from, the more surely were they going to have some chance of success.

GLEANINGS FAR AND NEAR.

1511. M. Molotoff himself is the finest type of revolutionary intellectual, his mind having been trained in the rigid logic of pure mathematics.—*Daily Telegraph*, November 15, 1940.

1512. Revealing father-draft figures for the first time, Selective Service disclosed that about 1,000,000,000 of the reclassified prewar dads had obtained occupational deferments in agriculture or other essential work.—*Washington Post*, December 19, 1943. [Per Mr. R. E. Beard.]

1513. The principle of rocket (or jet) propulsion is frequently misunderstood. Rockets do not operate by the pressure of escaping gas on the atmosphere, but by what is known as the principle of reaction. This can be illustrated if we assume a gas under pressure to be contained in a sealed cylinder. The cylinder then contains great energy forces pressing equally on its walls in all directions. If a vent is made at one end of the cylinder, the separate gas molecules will escape at high velocity under the effect of the pressure; and the fundamental physical law of "action and reaction" then comes into force, and whatever energy escapes at one end of the cylinder is equally exerted at the opposite end. If the energy is sufficient to overcome the weight of the cylinder, it moves forward.—*Birmingham Post*, June 25, 1945. [Per Mr. M. A. Porter.]

1514. THE CENTER OF BALANCE.—A popular method of finding the center of balance on the page is to divide the page into two parts, vertically, in accordance with the well-known rule of proportion, that "the smaller part is to the larger as the latter is to the whole". Balance, then, is determined by so dividing a page that the upper panel bears the same relation to the lower panel that the lower one does to the entire page. Therefore, if a page is divided into eight equal parts, the point located three units from the top and five units from the bottom is the center of balance on the page.—Polk, *The Practice of Printing*, Manual Arts Press, Peoria, Illinois (1926), p. 180. (The appropriate solution of the equation $x/(1-x)=(1-x)/1$ is $x=0{\cdot}382$, so that the rule given is an excellent approximation.) [Per Mr. F. W. Kellaway.]

1515. There are two sides to the life of every man, his individual life which is the more free the more abstract its interests, and his elemental swarm-life in which he inevitably obeys laws laid down for him.—Leo Tolstoy, *War and Peace*. [Per Mr. E. H. Lockwood.]

1516. Their flight will take them over the geographical North Pole and also over the magnetic North Pole—which is a little to the west.—*Daily Express*, May 15, 1945. [Per Mr. C. A. B. Smith.]

STATISTICAL SCIENCE AND AGRICULTURAL RESEARCH.

By D. J. Finney,

Lecturer in the Design and Analysis of Scientific Experiment, University of Oxford.

In a comparatively short paper, no complete survey of the range of agricultural problems whose solution involves statistical methods can be given. I have therefore chosen to confine myself to one particular theme, the development of the modern field experiment, even though this has meant that I must present a rather unbalanced picture on account of the omission of other important topics such as the use of sampling methods. During six years in the Statistical Department of the Rothamsted Experimental Station, I spent much time in the planning and analysis of a large annual programme of field experiments, and I have illustrated my paper with examples drawn from the work of this Station, at which many of the new ideas in field-experimentation were first tried. I have not attempted to discuss mathematical details at all fully, for the pure mathematics of statistics is a large subject in itself; my present concern is to show some of the practical applications of the theory.

In 1843, John Bennet Lawes, of Rothamsted House in Hertfordshire, began an experiment that is still in progress. On a field whose name, Broadbalk, has since become world-famous, he marked about 20 long strips of land of half an acre in area, and assigned each to a different manurial treatment. The following diagram shows the manuring of a portion of the field:

Plot No.	*Manuring*	*Mean yield 1852–1925 (cwt./acre)*
2B	Farmyard manure	19·3
3	None	6·7
5	Superphosphate, Sulphate of potash, and other minerals	7·8
6	As 5, + 2 cwt. Sulphate of ammonia per acre	12·5
7	As 5, + 4 cwt. Sulphate of ammonia per acre	17·6
8	As 5, + 6 cwt. Sulphate of ammonia per acre	20·1

The plots were sown with wheat; at the 1844 harvest, the plots were cut and threshed separately, and the yields under different treatments were compared. Since then wheat has been grown every year on Broadbalk, and, apart from minor modifications unimportant to the present discussion, the original manurial dressings of most plots have been repeated annually. What can be deduced from the results of a single year of this experiment? Very little; even adjacent areas of a field under uniform treatment may differ quite widely in yield, and consequently, in the experiment, differences apparently associated with the treatments may in fact be due to inherent differences in the yielding capacities of the plots. We might appeal to previous experience of the order of magnitude of yield differences between plots of this size treated identically, but we cannot be sure that such experience is applicable to *this* field in *this* year nor can we even measure the degree of uncertainty.

Results for 100 years—or even for 20—are more trustworthy than for only one, but, apart from the obvious disadvantages of having to wait 20 years,

even then the experiment does not answer very satisfactorily all the questions of interest. It provides useful data for a study of annual variations in yield under a series of different manurial treatments, but still does not provide reliable comparisons between average yields under different treatments. Plots 5, 6, 7, 8, for example, appear to indicate substantial benefits from nitrogenous manuring, which increase with increasing quantities of sulphate of ammonia. But this judgment assumes that the differences in yield are too large to be attributed to inherent differences in plot fertility. The four plots are arranged in systematic order, from the lowest to the highest level of nitrogen, and a fertility gradient across the field could produce a spurious appearance of a nitrogen effect. If nitrogen were having no effect of itself, such a gradient would probably be insufficient to produce a difference as large as that observed, but it might at least cause serious over- or under-estimation of the magnitude of a genuine nitrogen response.

The importance of the early field experiments, of which that on Broadbalk is an excellent example, both to agricultural practice and to the development of statistical thought, must not be neglected. Many of the results obtained in the first few years were so dramatic as to leave little doubt that plot differences could genuinely be ascribed to fertiliser effects. In 1920, R. A. Fisher was appointed as statistician at Rothamsted in order that he might analyse the accumulated data from the Broadbalk wheat and similar experiments, and in his analysis he first developed the powerful analysis of variance technique. By its use, he separated components of secular change in the yield of a plot from annual variation about this trend. He fitted polynomial equations to the data from each plot as a representation of the trend, and compared the forms of the curves for different manurial treatments. He also investigated the correlation of the residual annual deviations from the trend with measurements of meteorological conditions. For this purpose, he expressed the distribution of rainfall within each year by a fitted polynomial of the fifth degree, and took the coefficients of the polynomial as characterising the amount and seasonal distribution of rainfall in that year. These coefficients he then correlated with the yield residuals of any one plot, and derived a regression function which estimated the effect on yield of an inch of rainfall above or below average at any selected time of year. His highly ingenious technique has since been applied to many other data of like nature. In this particular experiment, it did not disclose a great deal of further information on the effects of the various fertiliser combinations on wheat yields ; though the effect of total rainfall was very considerable, and was related to the fertilisers given to the crop, within the limits of the available evidence the distribution of a specified total rainfall over the year did not appear to be of great importance.

But Fisher was not content merely to analyse the old data ; within a few years his introduction of the vital principles of *randomisation* and *replication* had revolutionised the whole technique of planning field experiments. Assignment of treatments to plots in a random order ensures that, though in a single experiment certain treatments may be favoured, the possibility of conscious or unconscious bias by the experimenter is removed. Replication, by providing two or more plots of each treatment, enables differences between mean yields for the various treatments to be compared with differences between plots treated alike, in order to assess whether they may be attributed to variations in plot fertility rather than to treatment effects.

The simplest of the new experimental designs is the *Randomised Block*. For this design, the experimental area is divided into a number of equal portions or *blocks*, each of which should be as homogeneous as possible in respect of soil. Each block is then subdivided into as many plots as there are

treatments to be tested, and to these a complete set of treatments is assigned in random order. The following diagram shows such an arrangement for four treatments, *A*, *B*, *C*, *D*, in five-fold replication :

I	*B*	*C*	*A*	*D*	*D*	*C*	*B*	*A*	IV
II	*A*	*C*	*D*	*B*	*C*	*B*	*A*	*D*	V
III	*A*	*D*	*C*	*B*					

If the yield of treatment j in block i is equated to an expression of the form

$$m + b_i + t_j + e_{ij},$$

where m is the general mean of the experiment, the b_i ($i = 1, 2, \ldots 5$) are constant in the blocks, and the t_j ($j = 1, 2, \ldots 4$) are constant over the treatments, then the constants may be determined so as to minimise $S(e^2)$. The structure of the design makes the b's and t's independent ; in fact, the b's are the deviations of block means from the general mean, and the t's are the deviations of the treatment means from the general mean. Hence differences between treatment means may be compared with the variation amongst yields of plots treated alike, after elimination of the effects of fertility differences between blocks.

Randomised blocks are frequently placed side by side in the field, thus enabling the experimenter to eliminate fertility differences in one direction. The *Latin Square* design enables fertility differences in two directions at right angles to be eliminated at once. " Latin square " is Euler's term for a square array of n^2 letters, n repetitions of each of n kinds, such that each letter occurs once in each row and once in each column :

A	*C*	*B*
C	*P*	*A*
B	*A*	*C*

C	*D*	*A*	*B*
A	*B*	*C*	*D*
D	*C*	*B*	*A*
B	*A*	*D*	*C*

If the space available for an experiment on n treatments can be divided into r rows of n plots, the treatments may be allocated according to the letters of r $n \times n$ Latin squares chosen at random from all Latin squares of that size.

One of the earliest Latin square experiments was a test of three different forms of potassic fertiliser for potatoes at Rothamsted in 1925. The arrangement of the plots and the yields, in units of 10 lb. per plot of one-fiftieth of an acre, were :

					Totals
	C 44	*D* 42	*A* 17	*B* 40	143
	A 28	*B* 44	*C* 42	*D* 41	155
	D 44	*C* 43	B 44	*A* 21	152
	B 45	*A* 24	*D* 41	*C* 39	149
Totals	161	153	144	141	599

The plots designated A received only nitrogenous and phosphatic fertilisers, whereas B, C, D received in addition dressings of sulphate of potash, muriate of potash, and potash manurial salts respectively, in amounts supplying equal weights of potassium. The total yields for the four treatments are:

Treatment	*Total*	*Tons per acre*
A	90	5·0
B	173	9·7
C	168	9·4
D	168	9·4
Standard error		±0·2

The yield of treatment k in row i and column j of the square may be equated to

$$m + r_i + c_j + t_k + e_{ijk},$$

where m is the general mean, and the r, c, t constants are estimated by minimising $S(e^2)$. The orthogonal properties of the Latin square make the equations for each set of constants independent of those for the other two sets; use of the principle of least squares for estimating the parameters assumes that the errors, e, of the individual plots are distributed according to the normal or Gaussian probability distribution

$$\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right) d\epsilon,$$

where σ is the same for all plots. In order to assess whether differences between yields under different manurial treatments can reasonably be ascribed to chance variations in plot fertility rather than to genuine manurial effects, the differences must be compared with the variation between plots under identical treatment after the elimination of positional effects associated with rows and columns. The sequence of calculations for this test can most conveniently be performed by finding the sum of the squares of the deviations of the 16 yields from the general mean and partitioning this into components representing the several sources of variation. This technique, introduced by Fisher, is known as the *Analysis of Variance* and is of great value in many kinds of statistical analysis.

Here the analysis of variance table is:

Source of variation	*Degrees of freedom*	*Sum of squares of deviations*	*Mean square*
Rows	3	20	7
Columns	3	62	21
Treatments	3	1194	398
Error	6	18	3·0
Total	15	1294	

The total sum of squares is the sum of the squared deviations of individuals from the general mean:

$$\begin{aligned} S(x-\bar{x})^2 &= Sx^2 - (Sx)^2/n \\ &= 44^2 + 42^2 + \ \ldots\ + 39^2 - 599^2/16 \\ &= 1294. \end{aligned}$$

Elementary theory of quadratic forms shows that this expression in the x's can be written as the sum of the squares of 15 linear functions of the x's, providing that the coefficients λ_i, μ_i, of any pair $S\lambda_i x_i$, $S\mu_i x_i$, satisfy the conditions

$$S\lambda_i = 0, \quad S\mu_i = 0, \quad S\lambda_i^2 = 1, \quad S\mu_i^2 = 1,$$

and the *orthogonality* condition

$$S\lambda_i\mu_i = 0.$$

The sum of squares is therefore said to have 15 *degrees of freedom*. The linear functions may be so chosen that three of them represent differences between rows (*i.e.* all plots in the same row have the same coefficient); the sum of squares for these 3 degrees of freedom may be shown to be

$$\tfrac{1}{4}(143^2 + 155^2 + 152^2 + 149^2) - 599^2/16 = 20,$$

the divisor 4 being the number of plots in each row total. Similarly, three more functions may be chosen to correspond with differences between columns, and three to correspond with differences between treatments; the sums of squares are

$$\tfrac{1}{4}(161^2 + 153^2 + 144^2 + 141^2) - 599^2/16 = \quad 62$$

and

$$\tfrac{1}{4}(90^2 + 173^2 + 168^2 + 168^2) - 599^2/16 = 1194$$

respectively. The sum of squares for the remaining 6 degrees of freedom is the minimal value of $S(e^2)$, but is most easily obtained by subtraction of the other components from the total.

The result of dividing this error sum of squares by its degrees of freedom is an estimate of σ^2, the variance (or squared standard deviation) per plot after elimination of row and column components. If the treatments are without effect, the mean square for treatments is also an estimate of σ^2. Fisher has shown that the probability distribution of the ratio of two independent estimates of variance of this kind, when each is derived from observations having a normal error distribution, is independent of σ^2 and is fully determined by two parameters, n_1 and n_2, the numbers of degrees of freedom for the two estimates. From tables that he has calculated, we may read that with $n_1 = 3$, $n_2 = 6$ there is a 5 per cent. probability that the ratio will exceed 4·76 and a 1 per cent. probability that it will exceed 9·78. We may now apply a test of significance for the effect of treatments. Briefly the argument is: "If the treatments did not affect the yield, a ratio of treatment mean square to error mean square in excess of 9·78 would be very unlikely to occur. The ratio in the experiment is 133. The possibility that this has arisen by chance when both mean squares are estimates of σ^2 may be neglected; we therefore reject this hypothesis and conclude that there are real differences between the yields for different treatments." The logic is a probability version of the familiar *reductio ad absurdum* argument. The estimated value of σ is $\sqrt{3{\cdot}0}$, which is 17 lb. per plot or 0·4 tons per acre, and the standard error of the mean yield for any one treatment is this quantity divided by the square root of the number of plots contributing to the mean. From the table of mean yields we see that all three forms of potash have produced very large increases in yield, but that differences between them are trivial by comparison with the standard error.

Since the introduction of randomised block and Latin square designs, many combinatorial problems have acquired a new interest. The enumeration of Latin squares is one of these; squares of any size can be constructed, but up to the present only those of side 7 or less have been counted, and, as no method other than exhaustive classification into types is available, further progress is likely to be slow. The total numbers of squares of various sizes are:

2×2:	2
3×3:	12
4×4:	576
5×5:	161,280
6×6:	812,851,200
7×7:	61,428,210,278,400

For some purposes we may wish to introduce a further orthogonal restriction into a Latin square; for example, four varieties (represented by Greek letters) might have been introduced into the experiment just discussed in such a way as to balance them over row, columns, and treatments:

$$\begin{array}{cccc} C\alpha & D\gamma & A\delta & B\beta \\ A\gamma & B\alpha & C\beta & D\delta \\ D\beta & C\delta & B\gamma & A\alpha \\ B\delta & A\beta & D\alpha & C\gamma \end{array}$$

Arrangements of this kind are known as Graeco-Latin squares; they exist for some but not all Latin squares of every size except those of the form $(4n+2)^2$. For 6×6 squares there are no solutions (Euler's famous 36 officers problem), and for 10×10, 14×14, ... there are conjectured to be no solutions.

Another type of design is the balanced incomplete block, which is analagous to the randomised block but differs in that not every treatment occurs in every block. For example, seven treatments may be arranged in seven blocks of three according to the columns of the following scheme:

$$\begin{array}{ccccccc} A & C & D & E & E & B & G \\ D & E & C & G & F & F & A \\ B & B & F & D & A & G & C \end{array}$$

Each treatment occurs the same number of times and each pair of treatments occurs together in the same number of blocks. If there are t treatments repeated r times, and arranged in b blocks of size k, then

$$rt = bk,$$

and

$$\lambda = r(k-1)/(t-1)$$

where λ is the number of times a pair of treatments occurs in the same block; in the example, $r=b=3$, $t=n=7$, $\lambda=1$. Not all sets of values of r, t, b, k satisfying these conditions and giving an integral value for λ have solutions. Some have been shown to give no solutions, and for a few, such as $r=7$, $t=15$, $b=21$, $k=5$, $\lambda=2$, it is still unknown whether or not solutions exist. The impossibility of certain arrangements is seldom a serious disadvantage to the experimenter, who is usually willing to add one or two extra treatments in order that a balanced design may be used. These designs are particularly useful for variety trials, in which many varieties of one crop, possibly thirty or more at a time, have to be compared, and the number of plots per block must be kept small in order to preserve reasonable soil homogeneity within each block.

The importance of varying one factor at a time, while all other relevant factors are held constant, used to be widely accepted as a principle of experimental science. In agricultural and biological research this is now generally recognised to be a mistaken and often a misleading policy, since optimal conditions for one factor may depend upon the conditions selected for the other factors. As an alternative, Fisher formulated the idea of *factorial experimentation*. He argued that if, for example, the response to nitrogenous fertiliser (i.e. the increase in yield produced by the fertiliser) may be affected by the amount of phosphatic fertiliser given to the crop, an experiment should contain different levels of application of nitrogen in combination with different levels of phosphate. The experiment will then provide an assessment of whether the responses to the two are inter-dependent and, if so, to what extent. If they are, that is an important practical point, and one which could never be discovered by separate experiments on nitrogen and phosphate; if not, nothing is lost, for the response to nitrogen can then be estimated as an average over all levels of phosphate and *vice versa*.

Frequently the experimenter is interested in a set of factors each of which is

to be tested at two levels only, perhaps presence and absence of some manurial constituent. For example, in each of the last few years about 20 experiments in various parts of the country have tested the value of nitrogen (n), phosphate (p), potash (k), and agricultural salt (s) for the manuring of sugar beet. Each experiment tested all the 16 combinations of plots without and with n, p, k, s, the rates of application being standard for the whole series. The treatment combinations may be represented by

$$1,\ n,\ p,\ np,\ k,\ nk,\ pk,\ npk,\ s,\ \ldots\ npks\ ;$$

the symbol " 1 " is conventionally used to indicate absence of all the fertilisers. Of the 15 degrees of freedom between these, one corresponds with the *main effect* of nitrogen, the difference between all plots with and all without n ; the appropriate difference may be written symbolically

$$(n-1)(p+1)(k+1)(s+1), \quad \text{denoted by } N.$$

Similarly, the main effect of phosphate may be written

$$(n+1)(p-1)(k+1)(s+1), \quad \text{denoted by } P.$$

The difference between two sets of treatment symbols obtained by expanding the expression

$$(n-1)(p-1)(k+1)(s+1), \quad \text{denoted by } NP,$$

measures the *interaction* of the effects of n and p, the difference between the response to n on plots with p and the response on plots without p. Three-factor interactions, of which

$$(n-1)(p-1)(k-1)(s+1), \quad \text{denoted by } NPK,$$

is typical, and a four-factor interaction

$$(n-1)(p-1)(k-1)(s-1), \quad \text{denoted by } NPKS,$$

may also be defined. There are in all 15 of these expressions ; they may be shown to be mutually orthogonal according to the definition given earlier, and from them 15 squares can be formed whose total is the sum of squares for treatments with 15 degrees of freedom. Thus the various main effects and interactions are estimated by independent expressions, and the significance and magnitude of each can be assessed independently of the others.

If this experiment were carried out in simple randomised blocks, each would contain 16 plots. So large a block is usually undesirable, as there may be considerable variations in soil fertility within its area. By the device of *confounding*, the block size may be reduced to eight. The $NPKS$ interaction, which is not likely to be of great size or practical importance, is measured by the difference between the two sets of treatments :

I	II
1	n
np	p
nk	k
pk	npk
ns	s
ps	nps
ks	nks
$npks$	pks

If two kinds of block are used, in equal numbers, the first containing the one set of treatments and the second the other set, the average difference between them will represent the $NPKS$ interaction ; hence the precision of estimation of this interaction will be low, as it will be affected by inter-block variation.

But all main effects and other interactions are measured by contrasts *within blocks*: each difference has four plots from each block taken positively and the other four negatively. Suppose that the experiment is to have two replicates. If there were no confounding, the analysis of variance would have the structure:

Source of variation	*Degrees of freedom*
Blocks	1
Treatments	15
Error	15
Total	31

but confounding modifies this to:

Source of variation	*Degrees of freedom*	
Blocks	3	(1 of them is $NPKS$)
Treatments	14	(all except $NPKS$)
Error	14	
Total	31	

The confounded experiment is likely to have a smaller error mean square than the unconfounded; consequently the main effects and other interactions (especially the interesting two-factor interactions) will be estimated with much greater precision, an advantage which more than outweighs the loss of information on $NPKS$.

By using four different kinds of block and confounding three treatment contrasts, the blocks could be further reduced to four plots each. We find that if we try to confound NPS as well as $NPKS$, then we are forced also to confound the main effect of K; a better choice is to confound, say, NPS and NKS, whereupon PK is also confounded. The four kinds of block are:

I	II	III	IV
1	n	p	k
npk	pk	nk	np
ns	s	nps	nks
pks	$npks$	ks	ps

The difference between I + II and III + IV is PK;
" " " I + III and II + IV is NKS;
" " " I + IV and II + III is NPS.

The analysis of variance table now takes the form:

Source of variation	*Degrees of freedom*	
Blocks	7	(including PK, NKS, NPS)
Treatments	12	(excluding PK, NKS, NPS)
Error	12	
Total	31	

The symbols for the 16 treatment combinations form a prime-power Abelian group, and the first type of block for each confounding scheme is a subgroup. The choice of degrees of freedom for confounding is restricted to the extent that the 15 symbols for treatment contrasts (the capital letters)

together with unity also form a prime-power Abelian group, and the contrasts confounded necessarily form a subgroup. Hence if NPS, NKS are chosen PK must also be confounded. These properties generalise very easily, and in an experiment with n factors each at two levels, the selection and construction of confounding arrangements can be carried out most expeditiously by reference to a prime-power group of order 2^n.

Experiments involving factors tested at more than two levels are often required. Providing that the number of levels is the same for all factors and is prime, the properties of prime-power groups can again be invoked for the construction of confounding schemes ; when these conditions are not satisfied, the problem is more complex though confounding can still be used. For example, a particularly useful experiment is that which tests three factors each at three levels. There are 27 treatment combinations, and these may be represented by the elements of a prime-power group of order 3 : $a^2b^0c^1$ or a^2c stands for the highest level of factor A, the lowest of B, and the middle level of C ; $a^0b^0c^0$ or simply 1 stands for the lowest level of all three factors. The 26 degrees of freedom between these combinations may be subdivided as follows :

Source of variation		*Degrees of freedom*
Main effects	A	2
	B	2
	C	2
2-factor interaction	AB	4
	AC	4
	BC	4
3-factor interaction	ABC	8
All treatments		26

The block size may be reduced from 27 plots to nine by confounding a pair of degrees of freedom out of the eight for ABC ; one of the four possible pairs leads to the following three types of block :

I	II	III
1	a	a^2
ab^2	a^2b^2	b^2
a^2b	b	ab
ac^2	a^2c^2	c^2
a^2c	c	ac
b^2c	ab^2c	a^2b^2c
bc^2	abc^2	a^2bc^2
abc	a^2bc	bc
$a^2b^2c^2$	b^2c^2	ab^2c^2

Again the first block is a subgroup of the treatment group, and, for reasons which cannot be discussed here, the confounded contrasts may be designated ABC, $A^2B^2C^2$, which with the addition of unity form a subgroup of the effects group.

Of course, a single experiment cannot possibly include tests of all the factors that might conceivably affect the crop. The experimenter is forced to select a set of factors of particular relevance to the problems in which he is interested, but the wide range of factorial designs now available enables him to make the most efficient use of the space and labour at his disposal. Often a factorial experiment need have no true replication of treatment combinations, as a standard error can be estimated by pooling the mean squares for a number of high-order interactions. Sometimes, indeed, a satisfactory experiment need

have only a fraction of the possible combinations of levels of the factors; for example, an experiment with eight factors at two levels involves consideration of 256 combinations of levels, but very useful information can be obtained by submitting only 128 of these to test in a trial on 128 plots. This procedure leads to some ambiguity of meaning for the treatment contrasts, since there are only 127 degrees of freedom available for the representation of 255 effects and interactions, but careful planning, again using the correspondence with the elements of Abelian groups, enables the ill consequence of this ambiguity to be minimised. The application of this device is limited, at least in agricultural work, but its possibility illustrates the immense gain in efficiency of modern experimentation by comparison with the traditional practice of " testing one factor at a time ".

In a short paper I have been able to discuss only one aspect of experimental design and analysis in agricultural research, namely field experimentation. The same principles are applicable to research in animal nutrition and to laboratory investigations, though details need to be varied according to the particular problem studied. Those who wish to know more of this subject will find much to interest them in *The Design of Experiments*, by R. A. Fisher; this book demands little previous knowledge of the technical mathematics, yet provides a valuable exposition both of the logical basis of the principles of good experimental planning and of the application of these to many research problems. D. J. F.

1517. Historical science in its endeavour to draw nearer to truth continually takes smaller and smaller units for examination. But however small the units it takes, we feel that to take any unit disconnected from others, or to assume a *beginning* of any phenomenon, or to say that the will of many men is expressed by the actions of any one historic personage, is in itself false.

It needs no critical exertion to reduce utterly to dust any deductions drawn from history. It is merely necessary to select some larger or smaller unit as the subject of observation—as criticism has every right to do, seeing that whatever unit history observes must always be arbitrarily selected.

Only by taking an infinitesimally small unit for observation (the differential of history, that is, the individual tendencies of men) and attaining to the art of integrating them (that is, finding the sum of these infinitesimals) can we hope to arrive at the laws of history.—Leo Tolstoy, *War and Peace*. [Per Mr. E. H. Lockwood.]

1518. Here, then, are the ingredients, the different elements for a balanced plan; the conditions to be aimed at, namely, one of equilibrium in which, while each pulls its due weight, it is not outweighed by any single one, is in no sense rigid. A plan must be ready and capable of continuous adjustment, like the polygon of forces which the engineer uses for calculating the stresses and strains in his structure, the shape may vary—it is not dictated by any formal pattern or design—but stability must always be reached.—From the " Preamble " to *A Plan for Plymouth* (page 5) by J. Paton Watson and Patrick Abercrombie. [Per Mr. F. W. Kellaway.]

1519. Arabic art always reminds me of mathematics. It is like algebra set to music, or, if you prefer, recurring decimals with wings on them. I experience the same unhappy feeling, when faced with Arabic art, that I would experience if you asked me to add up a long account. I feel that possibly only a senior wrangler could do justice to the exquisiteness of the Dome of the Rock.—H. V. Morton, *In the Steps of the Master*, Chap. III, § 2. [Per Mr. J. T. Combridge.]

THE COUNTERFEIT COIN PROBLEM.

By C. A. B. Smith.

There is a problem which is much in favour at present of which one version is this: a man has 12 pennies among which there may be a counterfeit coin, which can only be told apart by its weight being different from the others. How can one tell in not more than three weighings whether there is a counterfeit penny, if so which one it is, and whether it is heavier or lighter than a normal penny? (*Math. Gaz.*, XXIX (1945), pp. 227–229; XXX (1946), pp. 231–234.) This puzzle seems to have originated in America. The purpose of the present note is to make clear what is the best possible solution in the most general case of any version of the problem. No originality is claimed for these solutions, as no doubt many of them have been obtained independently many times before: but I am not aware that any complete and systematic account has yet been published.

The general problem and its various versions.

More generally we will suppose that the man has been given a set S of pennies. This set has been classified as follows: for some of the pennies it is known that they do not weigh less than the correct amount, but possibly more. For brevity these will be called "Above-weight" or "A" coins. Others will be known to weigh not more than the correct amount: these will be "Below-weight" or "B" coins. The third set are "Correct" or "C" coins. The fourth are "Doubtful" or "D": these may be of any weight. In the set S we shall suppose that there are $a(S)$ coins classified as A, $b(S)$ classified as B, $c(S)$ C's and $d(S)$ D's.

In what we shall call the "weak version no. 1" of the problem the man is now told that there is not more than one counterfeit coin, and he is allowed m weighings on an ordinary balance (using coins only, no weights) to find out (i) if there is a counterfeit, (ii) which it is, (iii) whether it is heavier or lighter than normal. Clearly at each weighing we must put an equal number of pennies on the two scales of the balance, for otherwise if the counterfeit was only slightly different in weight it would go undetected. Thus if there is no counterfeit the scales will balance at each weighing. So that in order to detect that a given penny is counterfeit it is necessary for it to be weighed at least once—*i.e.* no penny is left off the scales in every weighing.

Version 2 differs from version 1 in that it is now certain that there is an incorrect coin. We can then solve the problem in m weighings, even if we add a further coin of sort A or sort B which is never put on the scales.

In version 3 it is again certain that there is an incorrect coin, but it is only required to find which it is, not whether it is heavier or lighter than normal. In this case we can again have one more coin than version 1, and it may be any sort of coin. In view of these remarks it will clearly be sufficient to solve the version 1, since the solutions for the other versions follow immediately.

In the "strong" form of the problem (in any version) we make the additional restriction that it is necessary to state in advance exactly which coins are to be put on the scales at each weighing, the choice being uninfluenced by the results of the previous weighings. (For example, the man might have to leave the weighings to his small son, who was too young to follow any complicated instructions.) Clearly when there is a solution of the strong form there is also a solution of the weak form: surprisingly enough, the converse is also true except for a very small number of cases.

One obvious remark: to any version of the problem for which we have a solution we may add an unlimited number of C (correct) coins. These will not be weighed, and will not affect the solution in any way.

We shall use the following notation.

For any real number x, $[x+]$ denotes the smallest integer $\geqq x$, and $[x-]$ the greatest integer $\leqq x$. Thus $[3+]=[3-]=[2\frac{1}{2}+]=[3\frac{1}{2}-]=3$. If u is any integer, we can always write $u=2v+w$, where v, w are integers $\leqq [\frac{1}{2}u+]$.

Pennies of sorts A, B, and D will be called " questionable pennies ".

We shall write, for example,

$A+C$ for " an A coin + a C coin " : $A+A=2A$: $O=$ the set of no coins.

Clearly it is not possible to solve the problem for the combinations D, $2D$, $D+A$, $D+B$, $A+B$. These will be called the " insoluble combinations ", and we shall suppose that they do not occur. In all other cases we shall show that the problem is soluble in a finite number m of weighings, and determine the least value of m. Clearly also there is effective symmetry between the letters A and B, and so we shall tacitly assume throughout that any solution or result in which the letters A, B (or a, b) appear stands also for the corresponding result in which these two letters are interchanged.

Solution for weak form (version 1).

The idea of this solution is this : we use the information gained at each weighing to reclassify the coins, so that, for instance, a coin which was known to be D before the weighing might now be narrowed down to A (above-weight) or even C (correct). In that way, after m weighings we hope to whittle down the possibilities to a single penny, known to be counterfeit, or to show that all the coins are C.

It is most convenient to introduce for simplicity an extra coin E, which is never put on the scales : the idea is that we then always whittle down the possibilities to a single coin, but if the coin is E, then we know that in fact there is no counterfeit coin. In any set of coins S we define $e(S)$ to be the number of E coins in S, so that $e(S)=0$ or 1.

Now in any weighing we must do this : we must take the set S_A of above-weight pennies and split it up into three parts (some possibly empty), $S_A(R)$ which is put on the right-hand scale pan, $S_A(L)$ on the left-hand pan, and $S_A(N)$ on neither. Similarly for S_B, S_C, S_D, and S_E. If $a(R)$ denotes the number of coins in $S_A(R)$, *i.e.* the number of A pennies on the right-hand pan, and similarly for other cases, we must have :

$$a(R)+a(N)+a(L)=a(S) \text{ (similarly for } b, c, d), \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$e(R)=e(L)=0\ ;\ e(N)=e(S) \text{ (by definition of } E), \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$a(R)+b(R)+c(R)+d(R)=a(L)+b(L)+c(L)+d(L). \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

There are now three possibilities. Firstly, we may suppose that the right-hand pan is heavier (which we may denote by $R>L$). This may be due to either an above-weight penny, A or D, on the right-hand pan, or to a below-weight penny, B or D, on the left-hand pan. Accordingly we may now readjust our classification thus :

A pennies = those in $S_A(R)+S_D(R)$,
B pennies = those in $S_B(L)+S_D(L)$,
C pennies = all others.

If we write $a(R>L)$ for the number of pennies now classified A, we see

$$a(R>L)=a(R)+d(R),$$
$$b(R>L)=b(L)+d(L),$$
$$d(R>L)=e(R>L)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Secondly, however, we might find that it is the left-hand pan that is heavier ($R<L$). We then have by symmetry :

$$a(R<L)=a(L)+d(L),$$
$$b(R<L)=b(R)+d(R),$$
$$d(R<L)=e(R<L)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The third possible case is that the scales balance : then we know that all the coins on the scales are good ones, while those not on the scales retain their classification. Denoting this case by $R=L$, we have

$$a(R=L)=a(N), \text{ and similarly for } b, d, e. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Suppose now we write $\sigma(S)=a(S)+b(S)+2d(S)+e(S)$, then from equations (4), (5) and (6) we get

$$\sigma(R>L)+\sigma(R=L)+\sigma(R<L)=\sigma(S), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where the symbols have the obvious meanings. But the number σ has the following properties. Except for the uninteresting case in which we know that all the coins are C, we must have $\sigma>0$. In addition, when $\sigma=1$ we can see the solution of the problem at once without any weighings, while if $\sigma>1$ we can not. Thus the solution of the problem may be considered as the reduction of the value of σ to 1 in as small a number of moves as possible. Now since we cannot tell in advance whether $R>$, $=$, or $<L$, it follows that the greatest reduction in the value of σ that we can be certain of getting in one weighing is from $\sigma(S)$ to

$$\mu=\max\,[\sigma(R>L),\ \sigma(R=L),\ \sigma(R<L)]. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8$$

Now from (7) we see that $\mu+\mu+\mu\geqq\sigma(S)$, so that $\mu\geqq[\frac{1}{3}\sigma(S)+]$. By induction it follows that in m weighings we cannot be certain of reducing the value of σ to less than $[3^{-m}\sigma(S)+]$. Thus if we can get σ down to 1 in m weighings we must have $\sigma(S)\leqq 3^m$. This, however, is for the set containing the extra coin E: for the original set of coins (Z, say, so that $S=Z+E$) before the addition of E we must have :

Theorem 1. If the problem can be solved for a set Z in m weighings, then $\sigma(Z)\leqq 3^m-1$.

Conversely, if $\sigma(Z)\leqq 3^m-1$, then $\sigma(S)\leqq 3^m$, and we can certainly solve the problem in m or fewer weighings if we can be certain of reducing σ to a value not exceeding 3^{m-r} in r weighings. And that in turn will certainly be possible if we can reduce σ from its value, σ', say, before any weighing, to $[\frac{1}{3}\sigma'+]$ after that weighing. We must accordingly consider for what sets such a reduction is possible. Here is a list of some such sets, with the correct method of weighing, where a symbol such as U/V or W/X means that we weigh the set U on the left-hand pan, leave the set V or the set W off the balance, and put the set X on the right-hand pan.

List 1.

O/O or A or E/O.
A/O or B or $2B$ or $(B+E)$ or D or E/A.
A/O or B or E/C.
$2A/B$ or $(B+E)$ or D or $E/2A$.
$(A+B)/(A+B)$ or $(A+E)$ or $E/2D$.
C/O or B or E/D.
$A/2A$ or $(A+E)$ or B or $2B$ or $(B+E)$ or E/D.
$3A/(B+D)$ or $(D+E)/3A$.

D/O or A or $2A$ or $(A+B)$ or $(A+E)$ or E/D.
$2D/(A+3B)$ or $(A+2B+E)$ or $(3A+E)/2D$.
$4A/(B+D+E)/4A$.
$(A+B)/E$ or $(D+E)/(A+B)$.
$(A+B)/E/2C$.
$(A+2B)/(D+E)/(A+2B)$.
$(A+D)/A$ or $2A$ or $(2A+E)$ or $E/(A+D)$.
$(A+D)/D$ or $(D+E)/(B+D)$.

From these simple examples we can get an unlimited number of more complicated ones by adding any set of coins to one scale pan, and the same set to the other scale pan and also to the set of coins left off the balance—thus from the trivial case $O/E/O$ we may get the non-trivial one

$$6A+3D/6A+3D+E/6A+3D.$$

This operation does not alter the values of a, b, c, d, or e modulo 3, so that we may examine all possible cases, and we find that practically all of them are covered by the above list. Naturally the insoluble ones are not, and the other exceptional ones are the following, in which we can reduce σ from σ' to $[\frac{1}{3}\sigma'+1+]=\mu$, but not to $[\frac{1}{3}\sigma'+]$.

List 2.

$\sigma=3,\ \mu=2:\ A+B+C+E.$
$\sigma=5,\ \mu=3:\ 3A+B+E.$
$\sigma=6,\ \mu=3:\ (3A+B+D),\ (3A+D+E),\ (2A+B+D+E).$
$\sigma=9,\ \mu=4:\ 5A+B+D+E.$
$\sigma=6r+2,\ \mu=2r+2:\ (A+3rD+E),\ (3r+1)D.$
$\sigma=6r+3,\ \mu=2r+2:\ A+(3r+1)D,\ (3r+1)D+E.$

For example,

the correct way to weigh $A+B+C+E$ to give $\mu=2$ is $A/(C+E)/B$:

but because the coin E must always be left off the balance we can find no weighing which will reduce σ to 1.

Now, with the exception of $A+B+C+E$, none of these sets contains a C coin: and the set $A+B+C+E$ cannot result from any of the above weighings. (For, if it had resulted, since it contains the E coin, it must have been the set left off the balance: and there cannot have been any coins put on either of the two pans, or we would have had at least two C coins afterwards. But one would never weigh $O/A+B+C+E/O$, since $A/B+E/C$ would be better.)

Thus we see that, with the sole exception $A+B+C+E$, all sets containing one or more C coins may have their σ reduced to $\mu=[\frac{1}{3}\sigma+]$ in one weighing, and similarly in all subsequent weighings, and therefore if $\sigma\leqq 3^m$ they can be solved in at most m weighings. But after one weighing we certainly must have at least one C coin, as may be seen on examining List 1 (except for the insoluble sets, and certain sets soluble in one weighing). Hence in order that we may solve the problem in m weighings, it is necessary and sufficient that σ should be reduced to not more than 3^{m-1} in one weighing. And if $\sigma(S)\leqq 3^m$, that can certainly be done, except possibly for the sets of List 2. In fact, on examining that list we find the following failures, which are not soluble in m weighings, but are in $m+1$.

$$A+\tfrac{1}{2}(3^m-3)D+E,\quad \tfrac{1}{2}(3^m-1)D,\quad \tfrac{1}{2}(3^m-1)D+E,\quad A+\tfrac{1}{2}(3^m-1)D,$$

and when $m=1$, $A+B+C+E$, when $m=2$, $5A+B+D+E$.

Remembering that we started with a set Z, where $S = Z + E$, we obtain :

Theorem 2. The problem can be solved in m weighings for all sets Z for which $\sigma(Z) \leqslant 3^m - 1$, except for the insoluble sets and the following sets which need $m+1$ weighings :

$A + \frac{1}{2}(3^m - 3)D$, $\frac{1}{2}(3^m - 1)D$,

when $m = 1$, $A + B + C$, when $m = 2$, $5A + B + D$.

This theorem accordingly provides a complete solution for the weak form of the problem, in version 1. In accordance with the remarks we made earlier, we can readily deduce from this the complete solution for the other versions.

Solution for strong form (*Version* 1).

In this form of the problem we have to imagine we have a scheme stating beforehand exactly which coins are to be placed on each pan at each weighing. Now we can conveniently represent such a scheme in the following way. Firstly, we number the pennies off as $P_1, P_2, \ldots$, up to P_n (say). Then with each coin P_r we associate a vector $v(P_r)$ with m components,

$$v(P_r) = (v_1, v_2, \ldots v_m)$$

where the sth component v_s is defined to be 1, -1, or 0, according as the penny P_r is placed respectively in the right-hand pan, the left-hand pan, or neither. (For convenience we shall use an inverted figure 1 to denote -1, as in reverse notation.)

Because there must be an equal number of coins on the two scale pans at each weighing, we see that these vectors must satisfy

$$\sum_r v(P_r) = \underset{\sim}{0} = (000\ldots0). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Furthermore, the only information we have to show which penny is incorrect is the results of the weighings. If, for example, we find that the right-hand pan goes down in the first weighing, the left-hand one in the second, and neither in the third, we see that this must be due either to an above-weight coin which is placed on the right-hand pan in the first weighing, the left in the second, and neither in the third, and so has vector (1, 1, 0, ...), or else it may be due to a below-weight penny with vector (1, 1, 0, ...).

Thus, in order that the problem may be soluble, we must have the further conditions :

(i) If $v(P_r) = v(P_s)$, $r \neq s$, then either P_r and P_s are one A and one B, or at least one of them is C.

(ii) If $v(P_r) = -v(P_s)$, $r \neq s$, then either P_r and P_s are both A or both B, or at least one of them is C.

(iii) We do not introduce an extra coin E which is never weighed, so that no coin has vector $\underset{\sim}{0}$. ……(10)

Conversely, any set of vectors $v(P_r)$ satisfying the conditions (9) and (10) provides a solution of the problem in m weighings.

In order to help us find such solutions we notice that the $(3^m - 1)$ non-zero m-vectors may be divided into pairs of vectors $\pm v$ of opposite sign. We shall call these simply " pairs of vectors ", and the number $\frac{1}{2}(3^m - 1)$ of such pairs we shall denote by M. From condition (10) we see that in any set of vectors forming a solution we may choose from each pair either :

(i) one vector, associated with any coin, either A, B, or D (we do not introduce an extra E coin in the strong form), or

(ii) both vectors of the pair $\pm v$, associated with two A coins, or two B coins, or

(iii) one vector of the pair v, repeated twice; once associated with an A coin, and once with a B coin. In addition, we may associate with C coins any of the vectors without restriction, including the zero vector.

Suppose, then, that we can find a set of vectors $v_1, v_2, \ldots v_r$ (say), not more than one being chosen from each pair, whose sum is zero. Then we can easily construct from them a solution in this way: with each of the vectors $v_1, v_2, \ldots v_r$ we associate a coin (of any sort, but most usefully of sort D). With the remaining pairs $\pm v_{r+1}, \pm v_{r+2}, \cdots \pm v_M$ we may associate if we wish pairs of A coins or pairs of B coins, for such pairs of vectors have zero sum. We see that so far as solutions of this sort are possible, since we may have up to r D coins, and then up to $(2M-2r)$ A and B coins, we must have

$$\sigma = 2d + a + b \leqq 2r + (2M - 2r) = 2M = 3^m - 1,$$

the same condition as for the weak form. Thus we may expect roughly the same set of solutions for the strong as for the weak form: more exactly we have:

Theorem 3. For $r = 0$ and $3 \leqq r \leqq M - 1$, and for no other values of r, it is possible to find a set of r m-vectors $v_1, v_2, \ldots v_r$ satisfying the conditions:

(i) every component of every vector is 1, 0, or $\bar{1}$,

(ii) no $v_s = \underset{\sim}{0}$,

(iii) if $s \neq t$, then $v_s \neq \pm v_t$,

(iv) $\Sigma v_s = \underset{\sim}{0}$.

Proof. For $m = 1, 2, 3$, the theorem is readily verified directly.

Suppose, then, that it has been proved for $m = m' \geqq 3$. We wish to show that it follows for $m = m' + 1$. To do this we divide the range of r into 4 parts.

(I) If $3 \leqq r \leqq \frac{1}{2}(3^{m'} - 3)$, we simply take the solution for $m = m'$ and add on new component, identically 0.

(II) If $\frac{1}{2}(3^{m'} - 3) < r \leqq \frac{1}{2}(3^{m'+1} - 9)$, we write $r = 2g + h$, where g and $h \leqq \frac{1}{2}(3^{m'} - 3)$.

We take the solution for g m'-vectors, and add to it firstly an extra component 1, and secondly an extra component $\bar{1}$, thus obtaining a set of $2g(m'+1)$-vectors with zero sum. In addition, we take the solution for h m'-vectors and add to it the extra component 0. In all we have thus a set of $(2g + h) = r(m' + 1)$-vectors, which may be seen to form a solution under the above conditions.

(III) If $r = \frac{1}{2}(3^{m'+1} - 7)$ or $\frac{1}{2}(3^{m'+1} - 5)$, we write $r = 2g + h + 4$ where

$$g = \tfrac{1}{2}(3^{m'} - 5) \text{ and } h \leqq \tfrac{1}{2}(3^{m'} - 3).$$

Now we can find a solution for g m'-vectors: it will leave two pairs of m'-vectors unused, say $\pm\xi$ and $\pm\eta$. Then we can build up a solution thus we take:

(the solution for g m'-vectors with extra component 1);
(the same solution with extra component $\bar{1}$);
(the solution for h m'-vectors with extra component 0);
(the two vectors $\pm\xi$, with extra component 1);
(the two vectors $\pm\eta$, with extra component $\bar{1}$).

(IV) If $r=\frac{1}{2}(3^{m'+1}-3)$ we write $r=3g+3$ where $g=\frac{1}{2}(3^{m'}-3)$, we have a solution of g m'-vectors leaving out some pair, $\pm\xi$, say. We may then take as solution :

(the solution for g m'-vectors with extra component 1) ;
(the same solution with extra component 0) ;
(the same solution with extra component $\mathfrak{1}$) ;
(the vector ξ, with extra component 1) ;
(the vector $-\xi$, with extra component 0) ;
(the vector $\underset{\sim}{0}$, with extra component $\mathfrak{1}$).

This completes the induction.

We can see, moreover, that there can be no solution when $r=1, 2$; while when $r=\frac{1}{2}(3^m-1)=M$ there is no solution, because if we take any set of vectors obtained by choosing one from each of the M pairs, the sum of all these vectors must have all its components odd, and therefore cannot vanish.

The solution we obtain for $M-1$ vectors is particularly interesting : it was first sent to me by Mr. R. L. Brooks. It is this : we choose one vector from every pair except the pair $\pm(1\ 1\ 1\ \ldots\ 1)$ according to the rules :

(i) if all components preceding the first zero component are equal they are all 1's ;
(ii) otherwise the first component is $\mathfrak{1}$.

This type of solution is not, however, the only possible one. If $m\geqslant 3$, then for all values of r from 3 to M (but not for $r=1$ or 2) we can find solutions of the equation

$$v_1+v_2+\ldots+v_{r-1}+2v_r=\underset{\sim}{0},$$

where no two vectors belong to the same pair. This result may be proved in much the same way as Theorem 3 for $r=3$ to $M-1$, while we get the solution for $r=M$ from Brooks's solution of $\Sigma v_s=\underset{\sim}{0}$ for $(M-1)$-vectors by changing the vectors $(\mathfrak{1}\mathfrak{1}1\ \ldots\ 1)$, $(\mathfrak{1}1\mathfrak{1}\ \ldots\ \mathfrak{1})$, and $(\mathfrak{1}11\ \ldots\ 1)$ to $(11\mathfrak{1}\ \ldots\ \mathfrak{1})$, $(1\mathfrak{1}1\ \ldots\ 1)$, and $2(\mathfrak{1}111\ \ldots\ 1)$ and adding the vector $(\mathfrak{1}\mathfrak{1}\mathfrak{1}\mathfrak{1}\ \ldots\ \mathfrak{1})$—the resulting set of vectors still having zero sum. We may now associate the vectors $v_1, v_2, \ldots v_{r-1}$ with any $(r-1)$ coins whatever, the vector v_r with one A and also one B coin, and we may again add up to $(M-r)$ pairs of A and B coins.

Again, the equation $v_1+v_2+2v_3+2v_4=\underset{\sim}{0}$ is solved by $v_1=(110\ \ldots)$, $v_2=(1\mathfrak{1}0\ \ldots)$, $v_3=(\mathfrak{1}01\ \ldots)$, $v_4=(00\mathfrak{1}\ \ldots)$ when $m\geqslant 3$, and so provides a solution for such arrangements as $D+D+(A+B)+(A+B)+2\lambda A+2\mu B$.

We will now show how these arrangements give us the general solution. The cases $m=1, 2$ can easily be disposed of separately, so that we can confine ourselves to cases in which m, the number of weighings, is not less than 3. Consider first the situation when $d(Z)$, the number of D coins, is at least 2. We then see that we can solve the problem by assignments of vectors such as the following, where each A or D standing separately is given one vector from some pair, while in bracketed sums like $(A+B)$, A and B are both assigned the same vector.

(i) If $a(Z)$, $b(Z)$ are both even, solutions are

$$\begin{aligned} &qD && 3\leqslant q\leqslant M-1,\\ &D+D+(A+B)+(A+B),\\ &D+D+A+A,\\ &0, \end{aligned}$$

to any of which we may add pairs of vectors corresponding to $(A+A)$ or $(A+C)$ or $(C+D)$, provided that we do not use vectors from more than M pairs altogether.

(ii) If $a(Z)$ is odd and $b(Z)$ is even, solutions are

$$qD+A, \qquad 2\leqslant q\leqslant M-2$$

again with pairs $(A+A)$, etc., as before, and

$$(M-1)D+(A+C).$$

(iii) If $a(Z)$, $b(Z)$ are both odd, solutions are

$$qD+(A+B), \qquad 2\leqslant q\leqslant M-1,$$

again with pairs.

Since $\sigma(Z)=a(Z)+b(Z)+2d(Z)$, we see that these arrangements solve the problem in all cases for which $d(Z)\geqslant 2$, $\sigma(Z)\leqslant 2M=3^m-1$, except for the following ones :

List 3.

$$\left.\begin{array}{l} MD=\frac{1}{2}(3^m-1)D \\ (M-1)D+A \end{array}\right\} \text{ (as in weak form),}$$

$$2D+2(M-2)A,$$

$2D$ (insoluble).

Clearly all of these are insoluble in m weighings—although all except the last may be done in $(m+1)$. For example, a solution for MD would involve choosing M vectors, one from each pair, to sum to zero : and that we have seen to be impossible. (Theorem 3.)

If we investigate the other cases in which $d(Z)=0$ or 1, we find that, besides the insoluble ones, the following cases with $\sigma(Z)\leqslant 3^m-1$ cannot be solved in m moves.

List 4.

$$d(Z)=1, \quad c(Z)=0, \quad a(Z)+b(Z)=3^m-3\ ;$$

$$d(Z)=0, \quad c(Z)=0, \quad a(Z)+b(Z)=3^m-2\ ;$$

$$(3^m-4)A+D\ ;$$

$$(3^m-2)A+B\ ;$$

$$(3^m-2)A+B+C\ ;$$

and also when $m=2$ the following cases :

$$A+5B+D \text{ (as in weak form)}\ ;$$

$$3A+D\ ;$$

$$2A+2D\ ;$$

$$2A+2B+2D\ ;$$

$$A+B+3D.$$

So we see that there are a small number of cases for which $\sigma(Z)\leqslant 3^m-1$ which can be solved in the weak form but not in the strong form : but that in general the two forms agree : and, moreover, if we are given at least two correct coins, then all arrangements with $\sigma(Z)\leqslant 3^m-1$ are soluble in both forms in m weighings.

The cases in which all coins are of the same sort.

The most interesting cases are those in which all the coins are of the same sort, *i.e.* all A's, all B's, or all D's. Remembering what we said about the relations between the different versions of the problem, we see that if all the coins are of sort A (*i.e.* none below normal weight) or all of sort B (*i.e.* not above normal weight), then we can solve the problem in the weak form in m weighings,—

in version 1 (in which there need not be a dud) for any number of coins from 2 to 3^m-1, and

in versions 2 *and* 3 (in which there certainly is a dud) for any number of coins up to 3^m.

The strong form differs only in that in version 1 ($3^m - 2$) coins are not soluble in m weighings.

If all the coins are D, *i.e.* of unknown weight, then in either form we can solve the problem in m weighings,—

in versions 1 *and* 2 (in which we must say if the dud is heavy or light) for any number of coins from 3 to $\frac{1}{2}(3^m - 3)$, and

in version 3 (in which we merely find the dud) for any number of coins up to $\frac{1}{2}(3^m - 1)$, except 2. But if we are given in addition at least one correct coin, then we can solve the problem in versions 1 and 2 for any number of D's up to $\frac{1}{2}(3^m - 1)$, and in version 3 up to $\frac{1}{2}(3^m + 1)$.

The problem with weights.

In the above discussion we supposed that we were provided with a balance, but no weights. If we are provided with weights (so that we can determine the difference in weight between the two scale pans), and we know the weight of a correct penny, then the effect on the problem is similar to that of having an unlimited number of correct coins, except that we can now determine the exact weight of the counterfeit coin, and not merely whether it is heavier or lighter than normal.

Thus we can now solve the problem in m weighings in version 1 for *all* arrangements with $\sigma(Z) \leqslant 3^m - 1$, and corresponding results hold for versions 2 and 3.

If, however, we are allowed weights but do not know the weight of a correct coin, then the problem becomes very much more complicated, and I haven't yet obtained a general solution. However, one doesn't necessarily lose anything by not knowing the correct weight, as is shown by the fact that the following problem is soluble (I leave the solution to the reader):

"A man has 13 coins, not more than one of which is counterfeit. He marks them with the letters a, b, ... m, and hands them to his assistant to make three weighings, telling him exactly which coins to place on each pan in each weighing. From the three weights obtained he is able to tell the weight of a correct penny, whether there is an incorrect one, and if so which it is and how much it weighs. How can he do that?"

In conclusion, I would like to express my thanks to Dr. A. H. Stone for advice and suggestions regarding the paper.

CEDRIC A. B. SMITH.

1520. THE FOUR-DIMENSIONAL SPACE-TIME CONTINUUM. . . . This is the Space that is at this moment only present before our Eye, the only Space that was, or that will be, from Everlasting to Everlasting. This Moment Exhibits infinite Space, but there is a Space also wherein all Moments are infinitely Exhibited, and the Everlasting Duration of infinite Space is another Region and Room of Joys. Wherein all Ages appear together, all Occurrences stand up at once, and the innumerable and endless Myriads of yeers that were before the Creation, and will be after the World is ended, are objected as a clear and stable Object, whose several parts extended out at length, give an inward Infinity to this moment, and compose an Eternitie that is seen by all Comprehensors and Enjoyers.—(From the *Felicities of Thomas Traherne*, pub. Dobell, 1934, p. 104. Written about the middle of the 17th C.) [Per Mr. A. R. Pargeter.]

1521. Time really is nothing but a huge circle. You divide a circle of three hundred and sixty degrees into twenty-four hours, and you get fifteen degrees of arc that is the equivalent of each hour.—E. S. Gardner, *The case of the buried clock*, p. 82. [Per Mr. C. D. T. Owen.]

DOT, THE FAIRY: AN INTRODUCTION TO DECIMALS

(*for pupils aged* 8 *to* 10 *years*).

By R. S. Williamson.

Mary Rose was busy with her homework. John, her brother, had gone to the Pictures with some friends, and her parents were out as well. So Mary was quite alone. The shades of evening were falling, and the fire glowed cheerfully in the grate. But the growing gloom and the soothing warmth made Mary feel rather drowsy. She looked at her book with heavy eyes and read, "Multiply 2·763 by 100". Then she put down her pen with a sigh, and rested her head wearily on her hands.

Presently she was roused by hearing a quiet but tuneful voice. "Hullo! Little One," it seemed to say; and then, louder, "Hullo! Little One." She looked in the direction of the voice, and there was the teeniest, weeniest creature she had ever seen. She knew at once that it was a fairy, but she almost laughed aloud at being called "Little One" by someone as short as a match stalk.

"I am never short for long," said the fairy, to Mary's surprise. But she was much more surprised when she saw the fairy growing, taller and taller until Mary herself was much the shorter.

"Sometimes I long to be short," said the fairy. And immediately she began to grow short again. When she had reached her former height she addressed Mary again. "Well, Little One," said she. "And what is your name?" "Mary," was the answer, "Mary Rose."

"Mary, Mary, quite contrary," said the fairy. "Then you should feel quite at home here." She spoke so kindly that Mary timidly asked what she was called. "Oh! I am one of the Dorothies," said the fairy. "You can call me 'Dot', for short."

By this time Mary had been able to glance around. She saw scores of fairies like Dot. They were the Dorothies, she supposed. But what were they doing? There were arithmetical figures all over, and each fairy seemed to be hopping over them.

"Are the Dorothies playing hop-scotch?" she asked. "Playing," exclaimed Dot, "Certainly not; they are practising. But not hop-scotch; nor hop-welsh; nor butter-scotch. We call it 'La-mi-ced'. Alice knows all about that. You do not seem to be very contrari-wise."

This was all very puzzling to Mary.

"You will soon understand if you hold out your hands," said Dot. "Open them, with the palm upwards."

Mary did so. Almost immediately she felt a gentle touch on the left palm. Dot had hopped on to it.

"Keep quite steady," said Dot, as she made ready to hop again. She landed on Mary's right palm, and at once Mary began to grow. Up and up went her head, as taller and taller she became. Mary was very much afraid of becoming dizzy. "What an awful knock I shall get if I fall," she thought.

"So you long to be short," said the fairy, smiling.

"Yes, please," said Mary, rather meekly.

Back the fairy hopped to Mary's left palm. Mary at once stopped rising, and began to grow shorter. When she had reached her former height Dot hopped off her palm and on to the ground. Mary immediately stopped growing downwards. She was much relieved to find herself just the same Mary as before. In fact she was now quite pleased that she had had this adventure. But before she could thank the fairy, Dot spoke to her.

"Of course," said Dot, "you cannot really understand until you know the long and the short of it. So hold out your hands again." Mary did so, readily, and this time Dot hopped on to her right hand. "Steady," said the fairy, as she hopped from there on to Mary's left hand.

At once Mary felt herself growing shorter. But soon her courage began to grow less too. She feared that things around, which now seemed so large, would fall on her. She felt even the slight weight of the fairy, and was afraid lest she should drop her. "Growing taller was much easier," she said to herself. "That was like counting; you could go on and on and on, as long as you liked. But this is like counting backwards; you may start at 100, or 1000, or more, but you are certain to come to 0 in the end. What will happen if Dot does not stop me from growing shorter? There will be nothing of me left, not even enough for a funeral."

What other gloomy thoughts Mary would have had, nobody knows. For just at this moment her head gave a jerk, and there, before her eyes, lay her book. Mary stretched herself, and, feeling fully refreshed, took up her pen. She read the sum again, and then wrote, "$2{\cdot}763 \times 100 = 276{\cdot}3$."

"What wonderful things a small dot can do," she thought, as she put in the decimal point which turned 2 into 200. "Wonderful—Wonderland—Alice—Fairies." As these thoughts chased through her head she became so excited that she expressed the next thought quite loudly. "I know what I shall call the decimal point," she exclaimed. "I shall call it 'Dot, the Fairy'."

R. S. WILLIAMSON.

1522. HARMONIC PROGRESSION IN *The Times*.—During the recent war the Earned Income Allowance was reduced from one-sixth to one-tenth; in the 1946 Budget it is proposed to raise it to one-eighth. The comment in leading article in *The Times* of April 10th is: "The raising of the earned income allowance to a point half-way back to its 1939 level was widely expected." [Per Mr. J. T. Combridge.]

1523. William Henry Fox Talbot (1800–1877), another of the main and early contributors to the invention of photography. Fox Talbot was born in Lacock Abbey, Wiltshire, of a wealthy family, and was educated at Harrow and Trinity College, Cambridge. He became a well-known scholar and scientist, was a member of Parliament for 1832–1834 and, since 1831, in recognition of his mathematical work, a member of the Royal Society of London. His researches and experiments were fundamental for modern photography.—Lucia Moholy, *A hundred years of photography* (1939) p. 26.

Bedeutend weiter wie Brinkley und Wallace geht Talbot, der Erfinder der Uebertragung von Lichtbildern auf Papier, der gegenwärtigen Photographie, in zwei Abhandlungen "Researches on the Integral Calculus" (*Phil. Trans.* 1836, 177–215 u. 1837, 1–18). Die von Fagnano angewandten Rechnungen hat Talbot zu verallgemeinern und systematisiren gesucht. Es lässt sich aber nicht verkennen, dass die Abhandlungen über einige geistreiche Versuche nicht hinausgehen. Die gefundenen wesentlichsten Resultate und die Art ihrer Herleitung ist fast identisch wie bei Euler.—A. Enneper und F. Müller, *Elliptische Functionen* (zweite Auflage, 1890), p. 543.

1524. NONPLUSSED.—Announcement made at the Mathematical Association meeting in London to-day: "With the skill peculiar to mathematicians, we have collected rather more money for lunch tickets than we should have, and if anyone is a genuine 10s. short will he please apply for it?"—*Evening News*, April 24, 1946.

MATHEMATICAL NOTES.

1942. *Note on a paradox in Routh's "Analytical Dynamics".*

The following paradox arose from consideration of the problem of determining the frequency of oscillation of a loaded membrane. It is believed that readers will find the solution of the paradox entertaining and instructive.

In Routh's *Dynamics of a System of Rigid Bodies*, Part II, 6th Edition, 1930 (Macmillan), p. 458, the following problem is attempted.

"A uniform rectangular membrane, whose sides are a and b and mass M, has a particle of finite mass μ attached to it at a point whose coordinates are (h, k) when referred to the sides as axes. To find the motion."

Let T be the tension in the membrane, ρ the mass per unit area, and let $m^2 = T/\rho$.

Then Routh shows that the periods of oscillation are given by $2\pi/qm$ where

$$M/4\mu q^2 = \Sigma\Sigma \frac{\{\sin n\pi h/a, \sin n'\pi k/b\}^2}{\pi^2(n^2/a^2 + n'^2/b^2) - q^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

It will be seen that the double series is absolutely convergent. If the double series could be evaluated, we could deduce what mass μ should be placed at any given point so that one of the periods $2\pi/qm$ should have a given value.

Let q be less than π/b, and define ϕ by the relation

$$\pi^2 n'^2/b^2 - q^2 = \pi^2\phi^2/a^2.$$

Then equation (1) becomes

$$M\pi^2/4\mu a^2 q^2 = \Sigma\Sigma \frac{(\sin n\pi h/a)^2 (\sin n'\pi k/b)^2}{n^2 + \phi^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Routh then claims to have effected one of the two summations by making use of the relation

$$\frac{\sin^2 y}{1 - \xi^2} + \frac{\sin^2 2y}{2^2 - \xi^2} + \frac{\sin^2 3y}{3^2 - \xi^2} + \ldots = \pi \cdot \frac{\sin \xi y \sin \xi(\pi - y)}{2\xi \sin \pi\xi}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which holds for $0 \leqslant y \leqslant \pi$.

By making use of (3) when $\xi = \phi\sqrt{(-1)}$ and $y = \pi h/a$, Routh shows that equation (2) becomes

$$M\pi/\mu a^2 q^2 = \Sigma \frac{\{e^{y\phi} - e^{-y\phi}\}\{e^{(\pi-y)\phi} - e^{-(\pi-y)\phi}\}}{\phi\{e^{\pi\phi} - e^{-\pi\phi}\}} (\sin n'\pi k/b)^2 . \quad \ldots\ldots\ldots\ldots(4)$$

In the particular case when the load is placed on the medial line $h = \frac{1}{2}a$, we have $y = \frac{1}{2}\pi$ and equation (4) becomes

$$M\pi/\mu a^2 q^2 = \Sigma \left\{ \frac{1}{\phi} . \tanh \pi\phi/2 . (\sin n'\pi k/b)^2 \right\} . \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The above formulae are given explicitly in Routh, p. 459.

For simplicity, let us consider the particular case when the load is placed in the centre of the rectangle. Then $k = \frac{1}{2}b$, and equation (5) becomes

$$M\pi/\mu a^2 q^2 = \Sigma \frac{1}{\phi} . \tanh \pi\phi/2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

the summation being taken over odd numbers of n'.

However, the right-hand member may be shown to be divergent, so that apparently Routh has shown that a convergent series diverges!

We may present the paradox in a simpler form in the following manner. We know that the double series

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} 1/\{(2r+1)^2+(2s+1)^2\}$$

is absolutely convergent.

We now make use of the well-known expansion

$$\tan z = 8z \sum_{r=0}^{\infty} 1/\{(2r+1)^2\pi^2 - 4z^2\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This result is easily proved by contour integration, and appears as an example in Copson's *Functions of a Complex Variable* (Oxford, 1935), p. 157, No. 36.

We now substitute in (7) the particular value $z = i(2s+1)\pi/2$ where

$$i = \sqrt{(-1)}.$$

After rearranging terms we get

$$\sum_{r=0}^{\infty} 1/\{(2r+1)^2+(2s+1)^2\} = \pi/4(2s+1)\,.\,\tanh\,(2s+1)\,\pi/2. \quad \ldots\ldots\ldots\ldots(8)$$

If we sum equation (8) for all values of s we get

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} 1/\{(2r+1)^2+(2s+1)^2\} = \pi/4 \sum_{s=0}^{\infty} \frac{\tanh\,(2s+1)\pi/2}{(2s+1)}\,. \quad \ldots\ldots\ldots(9)$$

Consideration of the convergence of the right-hand member shows that

$$\frac{a_{s+1}}{a_s} = \left[\frac{2s+1}{2s+3}\right]\left[\frac{\tanh\,(2s+3)\pi/2}{\tanh\,(2s+1)\pi/2}\right] = 1-(1-\delta)/s + 0\left(\frac{1}{s^2}\right)$$

where $\delta \geqslant 0$, and by a well-known test the series is divergent, since $1-\delta \leqslant 1$.

Again it has been shown that an absolutely convergent series is divergent, but this time we cannot blame Routh! T. J. WILLMORE.

1943. *A propos de deux théorèmes de Faure.*

1. M. E. H. Neville (*Gazette*, XII, 1924, p. 23) a donné une preuve directe de cette propriété :

Soient F_1 et F_2 les foyers d'une ellipse, de grand axe $2p$, inscrite à un triangle d'orthocentre H. La quantité $4p^2 - HF_1^2 - HF_2^2$ est constante.

Utilisant ce théorème classique * : *La somme des carrés des demi-axes principaux d'une conique inscrite à un triangle, est égale à la puissance de son centre par rapport au cercle conjugué à ce triangle*, on arrive très vite à la même conclusion. En effet, O étant le centre du cercle circonscrit, de rayon R, au triangle fondamental ABC, p, q les demi-axes d'une ellipse inscrite de foyers F_1, F_2 et de centre ω, on a

$$p^2+q^2 = \omega H^2 - \tfrac{1}{2}(HO^2 - R^2),$$

et

$$\begin{aligned} HF_1^2 + HF_2^2 &= 2\omega H^2 + \tfrac{1}{2}F_1F_2^2 \\ &= 2(p^2+q^2) + HO^2 - R^2 + 2(p^2-q^2) \\ &= 4p^2 + HO^2 - R^2, \end{aligned}$$

ce qui justifie la proposition.

N.B.—La démonstration directe de M. E. H. Neville et celle de M. Kelly (*American Mathematical Monthly*, 1941, p. 338), ont pour avantage de conduire à celle du théorème invoqué de Faure dont les démonstrations élémentaires sont rares.

* *Nouv. Ann. de Math.*, 1861, p. 215. Des démonstrations ont, en outre, été données par Brisse, 1866, p. 257, et par P. Serret, 1865, p. 204.

2. On doit au géomètre français cet autre théorème * : *La somme des carrés des demi-axes principaux d'une quadrique inscrite à un tétraèdre orthocentrique est égale à la puissance de son centre par rapport à la sphère conjuguée au tétraèdre*, qui permet d'énoncer cette proposition : *Soient F_1 et F_2 les foyers d'un ellipsoïde de révolution, d'axes principaux $2p$ et $2q$, inscrit à un tétraèdre orthocentrique d'orthocentre H. La quantité $4p^2 + 2q^2 - HF_1^2 - HF_2^2$ est constante.*

Car, $2p$ étant l'axe focal de l'ellipsoïde de centre ω, inscrit à un tétraèdre orthocentrique $ABCD$ dont la sphère circonscrite, de rayon R, a pour centre O,

$$p^2 + 2q^2 = \omega H^2 - \tfrac{1}{3}(HO^2 - R^2) = \omega H^2 - (H_0).$$

Mais
$$\omega H^2 = \tfrac{1}{2}(HF_1^2 + HF_2^2) - \tfrac{1}{2}F_1F_2^2,$$
$$F_1F_2^2 = 4(p^2 - q^2);$$

donc
$$4p^2 + 2q^2 - HF_1^2 - HF_2^2 = (H_0) = \text{const.},$$

ce qui justifie la propriété en cause.

Cas particulier. Si l'ellipsoïde se réduit à l'une des sphères (I), de rayon r, tangentes aux quatre plans des faces du tétraèdre orthocentrique $ABCD$ on retrouve la relation

$$3r^2 + (H_0) = IH^2,$$

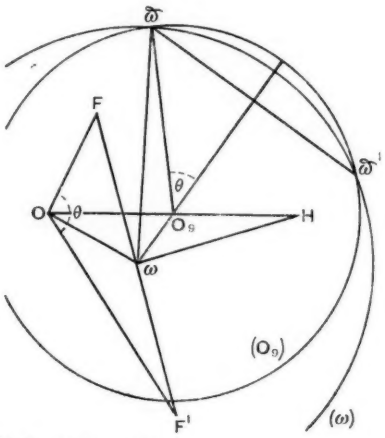
établie d'une autre manière par G. Fontené.†

V. THÉBAULT.

1944. *Sur un théorème de Weill-Aiyar.*

Soient : ABC un triangle, O le centre du cercle ABC, de rayon R, O_9 celui du cercle (O_9) des neuf points, H l'orthocentre, F et F' deux points conjugués isogonaux, ω le milieu de FF', α, β les demi-axes de la conique Σ inscrite au triangle et dont F et F' sont les foyers. Le théorème de Weill-Aiyar consiste alors en ce que ‡

$$OF \cdot OF' = 2R \cdot \omega O_9.$$



* *Nouv. Ann. de Math.*, 1866, p. 313.

† *Nouv. Ann. de Math.*, 1909, p. 66.

‡ Mathieu Weill, *Nouvelles Annales*, 1880, p. 253 ; W. Gallatly, *The modern geometry of the triangle*, p. 79.

On peut le démontrer aisément de la manière suivante. Dans le triangle FOF', on a

$$OF \,.\, OF' = (FF'^2 - OF^2 - OF'^2) : |\, 2\cos\theta\,|,$$

θ étant l'angle des diamètres OF, OF' du cercle ABC. D'autre part,

$$FF'^2 = 4(\alpha^2 - \beta^2),\quad OF^2 + OF'^2 = 2\,.\,O\omega^2 + 2(\alpha^2 - \beta^2);$$

d'où

$$OF \,.\, OF' = (\alpha^2 - \beta^2 - O\omega^2) : |\cos\theta\,|.$$

Mais

$$O\omega^2 = 2\omega O_9^2 - (\omega H^2 - \tfrac{1}{2}OH^2) = 2\omega O_9^2 - (\alpha^2 + \beta^2) - \tfrac{1}{2}R^2,$$

car *

$$\alpha^2 + \beta^2 = \omega H^2 - \tfrac{1}{2}(OH^2 - R^2).$$

Dès lors,

$$OF \,.\, OF' = 2\{(\alpha^2 - \tfrac{1}{4}R^2) - \omega O_9^2\} : |\cos\theta\,|. \qquad \text{.....(i)}$$

Le cercle podaire L de centre ω des points F et F', par rapport au triangle ABC, rencontre le cercle (O_9) en deux points réels (qui peuvent être confondus) qui coïncident avec les orthopôles ϖ et ϖ' des diamètres OF' et OF du cercle ABC et tels que

$$(O_9\varpi,\, O_9\varpi') = 2(OF,\, OF') = 2\theta,$$

l'angle (OF, OF') étant déterminé à un multiple de π près.†

En vertu du théorème classique relatif à la différence des carrés des côtés $\omega\varpi = \alpha$ et $O_9\varpi = \frac{1}{2}R$ dans le triangle $\omega\varpi O_9$, la relation (i) se transforme immédiatement en la suivante

$$OF \,.\, OF' = 2R \,.\, \omega O_9,$$

qu'il s'agissait d'établir. V. THÉBAULT.

1945. *The expression of $\sqrt{N}$ as a simple continued fraction.*

The following method of finding the quotients $a_1, a_2, \ldots$ when $\sqrt{N}$ is expressed as

$$m + \cfrac{1}{a_1 +} \cfrac{1}{a_2 +} \cdots$$

is simple and quick to use.

Put $\qquad N = m^2 + k \qquad (0 < k \leqslant 2m),$

$$2m = a_1 k + r_1 \qquad (0 \leqslant r_1 < k).$$

A convenient way of dealing with a numerical case is to set down the following columnar array :

$$\begin{array}{c|l} & 2m \\ a_1 & k \\ & r_1 \end{array}$$

in which k is placed below $2m$ and on the left of k is written a_1, the largest number of multiples of k contained in $2m$. Below k is written the remainder r_1 when $a_1 k$ is subtracted from $2m$.

At the side of this first array write down a new array, starting thus :

$$\begin{array}{|l} a_1 k \\ a_1 r_1 + 1 \end{array}$$

Dealing with $a_1 k$ and $a_1 r_1 + 1$ as we dealt with $2m$ and k in the first array,

* Faure, *Nouvelles Annales*, 1866, p. 308.

† G. Fontené, *Nouvelles Annales*, 1905, p. 506.

we obtain a quotient a_2 and a remainder r_2 which are written in the appropriate positions.

The whole picture up to this stage is now

$$\begin{array}{c|l} & 2m \\ a_1 & k \\ & r_1 \end{array} \qquad \begin{array}{c|l} & a_1k \\ a_2 & a_1r_1+1 \\ & r_2 \end{array}$$

The second array has been started one line below the first because we wish to regard the first element a_1k as derived from the element k, and the second element a_1r_1+1 as derived from the element r_1. The next stage will show this derivation more clearly.

The first two elements of the third array, written on the same level as a_1r_1+1 and r_2 respectively are obtained from a_1r_1+1 and r_2 by using a_2 as a multiplier and adding in elements from the first array. Before doing this we must obtain a fourth element of the first array, which is done by transferring a_2 to the space below a_1 and filling in $t=k-a_2r_1$ below r_1. The first two arrays now become

$$\begin{array}{c|l} & 2m \\ a_1 & k \\ a_2 & r_1 \\ & t_1 \end{array} \qquad \begin{array}{c|l} & a_1k \\ a_2 & a_1r_1+1 \\ & r_2 \end{array}$$

and the third array is started thus :

$$\begin{array}{|l} a_2(a_1r_1+1)+r_1 \\ a_2r_2+t_1 \end{array}$$

If a_3 and r_3 are formed from these first two elements of the third array in the same way as a_2 and r_2 were formed from the first two elements of the second array, we obtain the complete picture up to this stage as

$$\begin{array}{c|l} & 2m \\ a_1 & k \\ a_2 & r_1 \\ & t_1 \end{array} \qquad \begin{array}{c|l} & a_1k \\ a_2 & a_1r_1+1 \\ & r_2 \end{array} \qquad \begin{array}{c|l} & a_2(a_1r_1+1)+r_1 \\ a_3 & a_2r_2+t_1 \\ & r_3 \end{array}$$

For the sake of brevity, write the first entry of the nth array as A_n, the second as B_n, the third as C_n, and the fourth as D_n, so that

$$A_2=a_1B_1, \qquad A_3=a_2B_2+C_1,$$
$$B_2=a_1C_1+1, \qquad B_3=a_2C_2+D_1.$$

As before, a_3 is transferred to the space below a_2 and used to form D_2. Now form a fourth array by writing

$$A_4=a_3B_3+C_2,$$
$$B_4=a_3C_3+D_2,$$

and find the appropriate quotient a_4 and remainder r_4.

The process of forming arrays can be continued until $A_n=2m$, $B_n=k$ and the process then repeats.

The result is

$$\sqrt{N}=m+\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\ldots+\frac{1}{a_{n-1}+}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\,\ldots\,.$$

Example. $\sqrt{22}$. Here $m=4$, $k=6$. The working is shown below:

	8														
1	6		6												
2	2	2	3		8										
	2	4	0	4	2		8								
			3	2	0	2	3		6						
					2	1	2	1	6		8				
							1	8	0	8	1		8		
									6		0		6		

and the process repeats.

Thus
$$\sqrt{22}=4+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$$

The solution of the equation $Nq^2-p^2=\pm 1$ (p, q integral) can now quickly be obtained for any numerical value of N, because if

$$\sqrt{N}=m+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots+\frac{1}{a_{n-1}+}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots,$$

then
$$m+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots+\frac{1}{a_{n-2}}=\frac{p}{q}.$$

A short method for evaluating a continued fraction has been given in a previous note and consists effectively of working up one of the arrays occurring above, given the multipliers. In the case of the example $N=22$ above, the working is as follows:

	197
4	42
1	29
2	13
4	3
2	1
	1

so that $p=197$ and $q=42$, giving $22\,.\,42^2-197^2=-1$.

F. AYRES.

1946. *Theorems on conics.*

Here is a set of theorems on conics, though it is unlikely that they can be new, which include all possible cases—*i.e.* ellipse, parabola, hyperbola, circle, and two straight lines (parallel, intersecting, or coincident).

I. If two circles make double contact with any conic at P, P'; Q, Q', PP' being parallel to QQ', then (i) the length of the tangent from P or P' to the circle through Q, Q' = the length of the tangent from Q or Q' to the circle through P, $P'=ed=D/e$,

where d = distance between PP' and QQ',

D = distance between the centres of the circles,

and e = eccentricity of the conic (but see Notes).

(ii) If R is any other point on the conic, the sum or difference of the tangents from R to these two circles also $=ed=D/e$, and is therefore constant.

II. The locus of a point P which moves so that the length of the tangent from P to a fixed circle is proportional to the distance of P from a fixed straight line is a conic having double contact with the given circle at the points where it is cut by the given straight line (these points may, of course, be imaginary).

The eccentricity of the conic is the ratio of the length of the tangent to the distance from the straight line (but see Notes).

III. The locus of a point which moves so that the sum *or* difference of the tangents from this point to two fixed circles is constant is a conic having double contact with each of the circles (the points of contact may be imaginary).

The eccentricity of the conic is the ratio of the distance between the centres of the circles to the (constant) sum or difference between the tangents.

NOTES.—In all three theorems, the centre(s) of the circle or circles may lie on the minor axis of the conic if the conic is a hyperbola, in which case the eccentricity as defined in the Theorems is really the "second eccentricity" e', which is connected with the "true" eccentricity, e, by the equation

$$(1/e^2) + (1/e'^2) = 1,$$

or, with the usual notation, $(e')^2 = (a^2 + b^2)/b^2$.

Theorem I does not really include the case of two coincident straight lines, and in the case of a circle the two circles having double contact coincide with the circle itself, but all other cases are included.

Theorem II.

(i) When the eccentricity is 0, the locus is the given circle.

(ii) When the given straight line passes through the centre of the given circle, and the eccentricity is 1, the "parabola" becomes the two parallel tangents through the points of intersection.

(iii) The case of two intersecting straight lines is, of course, given by the tangents through the points of intersection of the given line and the given circle, provided the straight line does not pass through the centre of the circle and the suitable ratio is chosen.

(iv) The case of two coincident straight lines is given by the limiting case of a hyperbola whose "second eccentricity" is infinite.

Theorem III.

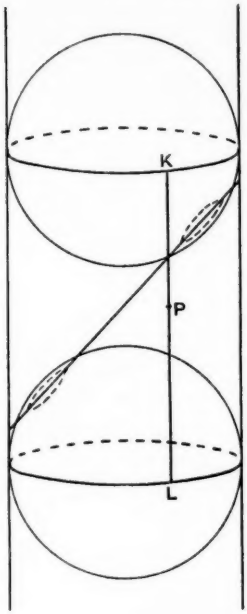
(i) If the two circles are concentric, the eccentricity is 0, and the locus is another concentric circle.

(ii) If the two circles have equal radii, and the eccentricity is 1, the locus is the two parallel common tangents.

(iii) For the more general case, the two common tangents, transverse or direct, are particular cases of the hyperbola, and the radical axis is the case of two coincident straight lines, so that this theorem really covers all cases, and it is interesting to note that we may have the sum constant for some points and the difference constant for other points *on the same conic.*

Theorems II and III. If the radius of any of the fixed circles is 0, the centre of the circle becomes a focus of the conic, and so we see that the well-known properties of a conic $SP = e \,.\, PM$, or $SP \pm S'P =$ constant are particular cases of the above two theorems.

To prove Theorem III, let P be a point on the surface of a cylinder or cone (the figure is drawn for the former case but the reader can easily see that the cone is a similar case). Let the generator through P touch two given spheres inscribed in the cylinder or cone at K, L. Then $PK \pm PL$ is constant. If P lies in a plane, the section of the cylinder or cone will be a conic and the sections of the two spheres will be two circles (which may be imaginary), and the lengths of the tangents from P to these circles will be equal to the lengths of the tangents from P to the two spheres, that is, PK, PL. Hence Theorem III. It is not difficult to prove Theorem II from this figure.



Similar results may be obtained in three dimensions, by replacing the circles by spheres and the conics by quadrics of revolution; in this case two parallel straight lines will be replaced by a cylinder and two intersecting straight lines by a cone. D. F. FERGUSON.

1947. *On Note* 1863.*

The example given in Note 1863 is an interesting illustration of the fact that the introduction of the line at infinity implies a departure from Euclidean geometry and that an unnatural strain is introduced by trying to make ordinary cartesian coordinates do the work of homogeneous coordinates. For Euclidean geometry the equation

$$\begin{vmatrix} 2x+3y-4 & 3x+7y-2 \\ 2x+3y+1 & 3x+7y-5 \end{vmatrix} = 0$$

is only a complicated way of writing $-21x-44y+22=0$ and represents a line. On the other hand, if we include the "line at infinity" in our geometry, the equation should be written in the form

$$\begin{vmatrix} 2x+3y-4z & 3x+7y-2z \\ 2x+3y+z & 3x+7y-5z \end{vmatrix} = 0,$$

and represents a conic, namely the pair of lines $z=0$ and $21x+44y-22z=0$.

* [A number of comments on Professor Anning's little joke have been received. Considerations of space prevent the publication of more than one. Ed.]

This may remind us of a group of similar difficulties which can easily be overlooked. They concern peculiarities of special cases. We usually say that if $S=0$, $S'=0$ are conics, $\lambda S+\mu S'=0$ gives the pencil through their four points of intersection. It is customary to discuss cases in which some of these points coincide. Are cases in which some are at infinity discussed? We usually say that any pencil contains two parabolas either of which may degenerate to a pair of parallel lines. We may point out that, if complex points are included in the geometry, cases such as that of both S, S' being rectangular hyperbolas present no exceptions, and explain the theorem: If a parabola is a rectangular hyperbola, it is also an equiangular spiral. But are we not liable to overlook the case of S, S' having parallel asymptotes? Here the two parabolas of the pencil coincide and are degenerate; and the degenerate form is not a pair of parallel or coincident lines, but is a pair of lines one of which is the "line at infinity". The determinant above is the equation of the "parabola" of such a pencil.

Another general statement is that the locus of the centres of a pencil of conics through four points is a conic. The equation

$$x^2+2\lambda xy+y^2-4x-4y+3=0$$

is the general equation of a conic through (1, 0), (3, 0), (0, 1), (0, 3). The equations for its centre are

$$x+\lambda y-2=0, \qquad \lambda x+y-2=0.$$

Incautious elimination gives $x-y=0$ as the locus of the centres. The question remains whether we should consider the points of the line $x+y-2=0$ to be part of the locus since each of them arises only as a centre of the single conic $(x+y-1)(x+y-3)=0$. R. C.

1948. *On Note* 1882.

Mr. R. C. Lyness and Mr. C. G. Paradine point out that if the cubes are

$$(6m_r+a_r)^3, \qquad (r=1, 2, 3, 4).$$

then their sum is $18p+\Sigma a_r^3$, where p is an integer. Since their sum is zero,

$$\Sigma a_r^3 \equiv 0 \pmod{18}.$$

This condition is fulfilled for all types except III, IV and IX, which are therefore impossible.

Mr. Paradine further remarks that for the types of partition of which solutions exist, a limited number of moduli in each case will yield linear or quadratic congruences which restrict the field of search.

For example, II (0, 1, 2, 3).

Application of the modulus 108 to the integers

$$6x,\ 6y+1,\ 6z+2,\ 6u+3$$

yields

$$y+4z+3u+2 \equiv 0 \pmod 6,$$

whence $$y+z \equiv 1 \pmod 3$$

and $$y+u \equiv 0 \pmod 2;$$

while from the modulus 72 we obtain

$$(4y+1)^2+(4u+1)^2 \equiv 18 \pmod{32}.$$

The odd quadratic residues of 32 are 1, 9, 17, 25, and we deduce the table (mod 8):

y	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
u	2	6	1	5	0	4	3	7	2	6	1	5	0	4	3	7

REVIEWS.

Eigenfunction Expansions Associated with Second-Order Differential Equations. By E. C. TITCHMARSH. Pp. 175. 20s. 1946. (Oxford University)

Let L denote a linear operator acting on a function $y(x)$. A function which satisfies the equation $Ly=\lambda y$, where λ is a constant, over a certain interval (a, b) and which also satisfies certain homogeneous boundary conditions at the ends of the interval, is called an eigenfunction of L, and the corresponding value of λ an eigenvalue. The general problem is to find the eigenfunctions associated with a given operator and then to expand an arbitrary function as a sum of eigenfunctions. The simplest case, when $L=d^2/dx^2$, occurs in the theory of the transverse vibrations of a violin string. Since the ends of the string are fixed, the boundary conditions are that $y=0$ when $x=0$ and $x=l$. It easily follows that the eigenvalues are $-n^2\pi^2/l^2$ where n is a positive integer, and that the corresponding eigenfunctions are $\sin n\pi x/l$. The eigenfunction-expansion reduces in this case to the Fourier half-range sine series. If, however, l is infinite, the expansion becomes the Fourier sine integral,

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin xt \int_0^\infty f(u)\sin tu\, du\, dt.$$

In the one case we have a discrete distribution of eigenvalues, in the other a continuous distribution.

Professor Titchmarsh's book deals with the eigenfunctions of the second-order differential operator

$$L=q(x)-\frac{d^2}{dx^2}.$$

When $q(x)$ is continuous in a finite interval (a, b), the theory goes back to the work of Sturm and Liouville more than 100 years ago. A discussion of the general case when the interval is infinite or when $q(x)$ is infinite at an end of a finite interval was first given by Weyl in 1910, who based his work on his theory of singular integral equations. An alternative method of dealing with the general case is by means of the theory of linear operators in Hilbert space, as developed by M. H. Stone in his well-known book. Those who, like myself, found the work of Weyl and Stone difficult, will welcome the new presentation developed by Professor Titchmarsh in a recent series of papers in the *Quarterly Journal of Mathematics* and amplified in the present book.

Titchmarsh arrives at the starting-point of his work by an interesting argument. He assumes that the partial differential equation

$$LV(x, t)=i\frac{\partial V(x, t)}{\partial t}$$

has a solution which satisfies the initial condition $V(x, 0)=f(x)$.

If one attempts to solve this by the complex Fourier integral

$$V(x, t)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty+i\alpha}^{\infty+i\alpha} F_+(x, \lambda)e^{-i\lambda t}d\lambda+\frac{1}{\sqrt{(2\pi)}}\int_{-\infty-i\alpha}^{\infty-i\alpha} F_-(x, \lambda)\, e^{-i\lambda t}\, d\lambda \quad \ldots(1)$$

where $\alpha>0$ and

$$F_+(x, \lambda)=\frac{1}{\sqrt{(2\pi)}}\int_0^\infty V(x, t)\, e^{i\lambda t}\, dt \qquad (\mathbf{I}(\lambda)>0)$$

$$F_-(x, \lambda)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^0 V(x, t)\, e^{i\lambda t}\, dt \qquad (\mathbf{I}(\lambda)>0),$$

one finds that

$$LF_+-\lambda F_+=-\frac{i}{\sqrt{(2\pi)}}f(x), \qquad LF_--\lambda F_-=\frac{i}{\sqrt{(2\pi)}}f(x).$$

If F_+ and F_- were found from these equations, we should then have, on putting $t=0$ in (1),

$$f(x)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty+ia}^{\infty+ia}F_+(x,\lambda)d\lambda+\frac{1}{\sqrt{(2\pi)}}\int_{-\infty-ia}^{\infty-ia}F_-(x,\lambda)\,d\lambda$$

$$=\sqrt{\frac{2}{\pi}}\,\mathbf{R}\int_{-\infty+ia}^{\infty+ia}F_+(x,\lambda)\,d\lambda,$$

since the two terms are conjugate complex numbers. Hence, writing

$$\sqrt{2\pi}F_+(x,\lambda)=i\Phi(x,\lambda),$$

we have

$$f(x)=-\frac{1}{\pi}\,\mathbf{I}\int_{-\infty+ia}^{\infty+ia}\Phi(x,\lambda)\,d\lambda$$

where

$$L\Phi-\lambda\Phi=-f(x). \quad \text{.....(2)}$$

Now let $\phi(x,\lambda)$, $\psi(x,\lambda)$ be two independent solutions of

$$Ly=\lambda y \quad \text{.....(3)}$$

which satisfy the boundary-condition at $x=a$ and at $x=b$ respectively; they cannot satisfy both boundary-conditions since λ is not necessarily an eigenvalue. If we choose the arbitrary constants in these solutions so that their Wronskian $W(\phi,\psi)=(\phi\psi'-\phi'\psi)$ is unity, the solution of (2) which satisfies the boundary conditions of $x=a$ and at $x=b$ is

$$\Phi(x,\lambda)=\psi(x,\lambda)\int_a^x\phi(y,\lambda)f(y)dy+\phi(x,\lambda)\int_x^b\psi(y,\lambda)f(y)\,dy \quad \text{.....(4)}$$

Professor Titchmarsh bases his whole investigation on the formula (4); integration of $\Phi(x,\lambda)$ round a large contour in the complex λ-plane gives the value $f(x)$, and the singularities of $\Phi(x,\lambda)$ on the real axis give a series or an integral expansion as the case may be. It is shown in Ch. I that this method gives a very simple solution of the classical Sturm-Liouville problem in which $q(x)$ is continuous on a finite interval.

In Ch. II, the discussion of the much more interesting singular case is started. The interval is taken to be $(0,\infty)$, and $q(x)$ is supposed to be continuous in every finite interval $(0,X)$. Two independent solutions θ, ϕ of (3) are constructed, such that

$$\phi=\sin\alpha,\ \phi'=-\cos\alpha\,;\qquad \theta=\cos\alpha,\ \theta'=\sin\alpha$$

when $x=0$. The solution $y=\theta+l\phi$, where l is a constant, satisfies the boundary-condition $y\cos\beta+y'\sin\beta=0$ when $x=b$ if

$$l=-\frac{\theta(b)\cot\beta+\theta'(b)}{\phi(b)\cot\beta+\phi'(b)}\,.$$

For each fixed value of b, the point l describes a circle C_b in the complex plane as β varies from 0 to π; moreover, if $\mathbf{I}\,\lambda\neq0$, C_b lies inside C_c if $b>c$. It then follows, as Weyl showed, that C_b converges either to a limit-point or to a limit-circle as $b\to+\infty$.

If $m(\lambda)$ is the limit-point or a point on the limit-circle, the solution

$$\psi(x,\lambda)=\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

belongs to $L^2(0,\infty)$ when $\mathbf{I}\,\lambda\neq0$. In the limit-circle case, every solution of (3) belongs to $L^2(0,\infty)$, in the limit-point case only $\psi(x,\lambda)$ does. The function $m(\lambda)$ is an analytic function regular in the upper and lower half-planes; the nature of the eigenfunction expansion depends on its behaviour on the real axis.

In the simplest case, $m(\lambda)$ has no singularities but simple poles $\lambda_0, \lambda_1, \lambda_2, \ldots$ on the real axis. If the residue at λ_n is r_n, the functions

$$\psi_n(x) = \sqrt{r_n}\phi(x, \lambda_n)$$

form a normalised orthogonal set. The main theorems of Ch. II for this case are as follows :

I. *If* $f(x)$ *is* $L^2(0, \infty)$

$$\int_0^\infty \{f(x)\}^2 dx = \sum_0^\infty c_n^2$$

where

$$c_n = \int_0^\infty f(x)\,\psi_n(x)\,dx$$

II. *If* $f(x)$ *and* $L\,f(x)$ *are* $L^2(0, \infty)$, *if* $f(0)\cos\alpha + f'(0)\sin\alpha = 0$, *and if* $W\{\psi, f\} \to 0$ as $x \to +\infty$ *for every non-real* λ, *then*

$$f(x) = \sum_0^\infty c_n \psi_n(x), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

the series being uniformly and absolutely convergent in every finite interval. The rather heavy restrictions on $f(x)$ in Theorem II are relaxed in Ch. IX and Ch. X at the expense of $q(x)$.

In Ch. III the general singular case is taken up. All that is known about $m(\lambda)$ is that it is regular in $\mathbf{I}\lambda > 0$ where $\mathbf{I}\,m(\lambda) < 0$. It is then proved that

$$\lim_{\delta \to +0} \int_0^\lambda \mathbf{I}m(u + i\delta)\,du = -k(\lambda)$$

where $k(\lambda)$ is a non-decreasing function of the real variable λ. In the special case of Ch. II, where $m(\lambda)$ is assumed to be meromorphic, $k(\lambda)$ is a step-function. In the general case, we have the following expansion theorem :—

III. *If* $f(x)$ *satisfies the conditions of Theorem II,*

if

$$\chi(x, \lambda) = \int_k^\lambda \phi(x, u)\,dk(u), \qquad g_1(\lambda) = \int_0^\infty \chi(y, \lambda) f(y)\,dy,$$

then

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x, u)\,dg_1(u). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The integrals in III are Stieltjes integrals. An interval in which $k(\lambda)$ is constant contributes nothing to the representation (7). An interval in which $k(\lambda)$ increases continuously contributes an integral-term, a point of discontinuity contributes a single term. In particular, if $k(\lambda)$ is a step-function, (7) reduces to (6). The spectrum of L, *i.e.* the set of eigenvalues of L, is then the complement of the set of points in the neighbourhood of which $k(\lambda)$ is constant.

The nature of the spectrum is discussed in Ch. V. Broadly speaking, there are four different cases :

(*a*) If $q(x) \to +\infty$ as $x \to +\infty$, there is a point-spectrum.

(*b*) If $q(x) \to 0$, there is a point-spectrum (possibly null) in $(-\infty, 0)$ and a continuous spectrum in $(0, \infty)$.

(*c*) If $q(x) \to -\infty$ and $\int^\infty |q(x)|^{-1/2} dx$ is divergent, there is a continuous spectrum.

(*d*) If $q(x) \to -\infty$ and $\int^\infty |q(x)|^{-1/2} dx$ is convergent, there is a continuous spectrum in $(-\infty, 0)$ and a point-spectrum (possibly null) in $(0, \infty)$.

The whole theory is well-illustrated in Ch. IV where it is used to obtain representations of an arbitrary function as a series of Legendre, Bessel, Laguerre or Hermite functions or as a Fourier, Hankel or Weber integral.

The book is described on the wrapper as "definitely for mathematicians", and it is a difficult one at that; for I found that I had to read it at my desk with pencil and paper. This is due to the nature of the subject, and I personally am truly grateful to Professor Titchmarsh for his brilliant new presentation of a most fascinating subject. It is, perhaps, too late to complain of the hybrid words "eigenfunction" and "eigenvalue" when "latent function" and "latent root" sound so much better; one can only be thankful that the late Professor Bateman's abbreviations "eif" and "eit" never found favour.

It is unnecessary to make any comment on the high quality of the printing, as the book comes from the Clarendon Press. E. T. COPSON.

Analytical method in dynamics. By H. O. NEWBOULT. Pp. 81. 7s. 6d. 1946. (Oxford University Press)

This book is intended to serve as a supplement to the existing textbooks on dynamics and as such it should be very useful to students. Starting with a chapter on two-dimensional dynamics, it passes on to four chapters on three-dimensional dynamics and includes a chapter on the motion of an inextensible flexible chain in its plane. The book deals mainly with the use of moving axes and employs vector methods as well as those involving coordinates.

Teachers of dynamics will have their own ways of dealing with the difficulties of moving axes and a reviewer can probably best fulfil his task by confining himself to a few remarks, based on his own experience, which his reading of the book has suggested to him.

1. The phrase "velocity relative to O", where O is the origin of the moving axes, is not precise enough and may be misleading to students. I prefer to use "velocity referred to moving (or fixed) axes at O". For example, in two-dimensional motion with a moving origin O and rotating axes at O, the components of the velocity, *referred* to specified moving axes at O, of a particle at P (x, y) are $u_0+\dot{x}-y\omega$ and $v_0+\dot{y}+x\omega$, where (u_0, v_0) is the velocity of O, *referred* to fixed axes in the plane of the motion, and the velocity of P *relative* to the moving axes at O is $(\dot{x}, \dot{y})$, the moving and fixed axes being parallel at the instant considered. It is desirable, at any rate at an early stage of the treatment, to use unambiguous phraseology even if it involves more lengthy statements.

2. The idea of a "convection velocity" (*vitesse d'entraînement* in French textbooks) is a very useful one. In the above notation $(u_0-y\omega, v_0+x\omega)$ is the convection velocity of P, being the velocity of P supposed rigidly connected to the moving axes which constitute a rigid lamina, and then the velocity of P referred to the moving axes at O is the sum of its velocity $(\dot{x}, \dot{y})$ relative to the moving axes at O and its convection velocity. This provides a simple and satisfactory proof of the theorem that the motion of a lamina in its own plane can be represented by the rolling of its body-centrode on its space-centrode. If I_b is the point of the lamina which is its instantaneous centre at a given instant and I_s is the point in the plane of the motion which coincides with I_b at the instant then, since the velocity of the point I $(=I_s=I_b)$ of the lamina is equal to the sum of its velocity relative to the lamina and its convection velocity, and the latter is zero, it follows that the velocity of I referred to fixed axes in the plane of the motion is equal to its velocity relative to the lamina. Hence these velocities have the same direction and the same magnitude, so that the space- and body-centrodes touch and the arcs of these two curves measured from any initial positions of I are the same. This is the proof as given in French textbooks, which should be consulted for a more detailed exposition.

3. The treatment of the motion of a rigid body in three dimensions, one of its points O being fixed, as given in Chap. II of the book, is that used by Appell, *Traité de Mécanique Rationnelle*, Vol. I, Chap. 2. This proof can be put into a vector form in which, if (α, β, γ) are unit vectors along the moving rectangular axes $(Oxyz)$ fixed in the body, the angular velocity Ω of these axes referred to fixed axes $(OXYZ)$ in space appears in the symmetrical form as

$$\alpha(\dot{\beta}\gamma) + \beta(\dot{\gamma}\alpha) + \gamma(\dot{\alpha}\beta)$$

so that its components ω_1, ω_2, ω_3 referred to the moving axes are $(\dot{\beta}\gamma)$, $(\dot{\gamma}\alpha)$, $(\dot{\alpha}\beta)$ in agreement with the results of the analytical treatment given in the book. This vector proof appears to be due to T. Levi-Civita and U. Amaldi, *Lezioni di Meccanica Razionale* (2nd edition, 1930), Vol. I, pp. 180–183, and I owe my knowledge of it to Dr. V. C. Ferraro of King's College, London.*

4. In the case of the motion of a rigid body referred to moving axes, one of which (the third) is fixed in the body, it is required to prove that $\omega_1 = \theta_1$, $\omega_2 = \theta_2$ where $(\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body and $(\theta_1, \theta_2, \theta_3)$ that of the moving axes. In order to obtain the result (Chap. III, p. 50) it is simpler to consider any point on the third axis at a distance c from the common origin of the axes fixed in the body and the moving axes; in this case its velocity is $(c\omega_2, -c\omega_1, 0)$ as a point of the body and $(c\theta_2, -c\theta_1, 0)$ as a point in the third axis of the moving set. Since this third axis is fixed in the body it follows that $\omega_2 = \theta_2$ and $\omega_1 = \theta_1$.

Apart from the above remarks, of which only the first can be considered as a criticism, the treatment given in the book is in my opinion, instructive and interesting and deals with many points which are as a rule insufficiently stressed in other textbooks. This is particularly true of the chapter on two-dimensional dynamics which should be read by all who are interested in the teaching of dynamics. The treatment by vectors of the three-dimensional problems in articles 3·3, 3·6 and 4·2 will be welcomed by the increasing number of students and teachers who make use of vectors.

H. R. Hassé.

Relaxation Methods in Theoretical Physics. By R. V. Southwell. Pp. vi, 248. 20s. 1946. (Oxford University Press)

In 1935 there appeared papers† by Professor Southwell (as he then was) describing a method of determining forces and displacements in loaded frames by the "systematic relaxation of constraints". These papers contained the germ, as well as the title and much of the terminology, of the now well-known "relaxation" methods. In 1940 progress in what may be loosely specified as one-dimensional problems was recorded and systematised in a book‡, but it was by then clear that a wide range of two-dimensional problems also lay within the scope of these methods. A second book was promised, describing these developments, when sufficient material had been harvested from the fields which were then already being diligently ploughed by Southwell and his research team. This promise has been redeemed in the book under review. It is concerned with the numerical solution of boundary value problems for second order partial differential equations of elliptic type in two independent variables. In it (and in papers already published) a third volume, dealing with biharmonic analysis, is foreshadowed.

* I find that this proof is essentially that given in Gibbs and Wilson, *Vector Analysis* (1907), Art. 60, p. 131.

† *Proc. Roy. Soc. A*, 151, pp. 56–95 and 153, pp. 41–76.

‡ *Relaxation Methods in Engineering Science* (Oxford University Press). Reviewed in *Math. Gaz.*, XXV, pp. 180–182, 1941.

For numerical work a continuous problem must be replaced by a discrete one, and here the general aim is to determine the values of a wanted function at the nodal points of a lattice. Hexagonal, square, and equilateral triangular lattices are considered, but the first named is scarcely used. The differential equations are replaced by their finite difference equivalents, while finite difference or ordinate formulae for numerical integration and differentiation, and for interpolation, will evidently be needed. These fundamental preliminaries provide the material of Chapter I, which also includes a spirited defence of the method. More important (for we would have thought that the necessity and practical adequacy of the determination of functions at lattice points only would have been generally admitted) is the author's insistence upon the inevitability of uncertainties in the *specification* of any practical problem, i.e. of " errors " in the data, and of the logical absurdity of insisting upon accuracy in the solution greater than these uncertainties warrant.

In Chapter II we are introduced to the " relaxation " way of thought, and in Chapter III to the " relaxation " technique, as applied to plane potential problems. Southwell's thinking has been largely done (or at least expounded) in terms of mechanical analogies. Although they are not necessary (in the sense that the methods are capable of formulation *now* in abstract mathematical terms), and indeed are not slavishly adhered to, these analogies both serve as a guide in the attack upon a new problem, and in the actual computation, and give rise to the terminology ; it is, in fact, difficult to believe that the methods could have been invented and developed *without* their aid.

Poisson's equation in two dimensions (x, y)

$$\nabla^2\phi + Z(x, y) = 0 \qquad (\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2)$$

has an analogy in the transverse displacement w of a membrane stretched by a tension T and subject to a transverse load (pressure) of intensity $p(x, y)$,

$$\nabla^2 w + p(x, y)/T = 0.$$

To represent the corresponding finite difference equation the membrane is replaced by a tensioned net and the transverse pressure is concentrated into forces applied to the nodes. If *arbitrary* displacements are given to the nodes, the resultant transverse force at any node, due partly to the applied forces and partly to the transverse components of the tension in the strings meeting at the node, will vanish (and the node be in equilibrium without the action of the constraint which maintains at that point the arbitrary deflection) only if the finite difference equation (which now acquires a complete physical analogy) is there satisfied. Where it is not satisfied, the constraint must exert a " residual force ", whose value can be computed from the displacement at the node and those at the (four or six) adjacent nodes. By altering the displacement at a node (" relaxing a constraint ") some or all of the " residual force " may be transferred, in readily calculated amounts, to the adjacent nodes. Starting with an approximate (guessed) set of displacements one can, by " relaxing " node after node in succession, " liquidate " the residuals, or at least reduce them all to less than some quantity of the order of the uncertainty in the data or the error of the finite difference approximation. The errors are as it were, brushed off the field, over the boundary—for there are, in addition to the governing differential equation, boundary conditions to be satisfied, the simplest case being where the values of ϕ (displacement, in the analogy) are specified there.

It is the essence of the relaxation technique that, although the wanted function is the goal, and although an estimate of its approximate values forms a necessary starting-point, the *computation* process concerns itself with " residuals " (errors in the governing equation) and the effect upon these of

changes (only) in the wanted function. Not until the residuals have been liquidated is stock taken in the net change, and the closer approximation to the wanted function found.

The accuracy of the finite difference approximation evidently depends on the size of the interval—the mesh-length of the net. In the early stages a coarse mesh is used, but as the original crude guesses are refined, advance is made to successively finer nets, until it is deemed that further refinement is unnecessary.

Types of boundary condition other than those specifying values of the function, and cases where the boundary passes between (i.e. not through) nodes, so making "incomplete stars", are brought within the scope of the method, and there are devices for speeding up the liquidation process.

Chapter IV applies the methods to the conformal representation of a given region upon some standard region, a process necessary in some cases for its own sake, and desirable in others as a preliminary step enabling the relaxation process to be carried out over the transformed region in which the boundary passes through nodal points (so that the complications of incomplete stars are avoided).

Chapter V extends the methods to the "quasi-plane-harmonic" equation

$$\frac{\partial}{\partial x}\left(\chi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\chi\frac{\partial\phi}{\partial y}\right)+Z(x,y)=0$$

in which χ may either be known at the outset as a function of position, or is a function of ϕ and/or its first derivatives; in the latter case the equation is non-linear, although this fact only complicates the calculations without altering the general principles.

Finally, in Chapter VI, a number of problems which seem so far to have defied "orthodox" (formal) analysis, are solved, problems in which a double condition is prescribed along a boundary whose position is initially unknown, and whose determination is an essential element in the problem. Such problems concern plastic and elastic regions in the torsion problem, wet and dry regions in the problem of percolation of water through porous materials, and the hydrodynamic problem of free streamlines with the effect of gravity included.

The book ends with tables for numerical differentiation and integration, a list of original papers, and an index to the 35 specific problems whose solution is given in detail, as well as name and subject indices.

The book is frankly didactic, and although it is difficult for one who has followed closely the gradual developments of the methods to put himself in the place of one meeting them for the first time, the purpose would seem to be fully achieved. It seems clear from internal evidence that the author must have taken great pains to ensure that his exposition should be orderly, clear, and complete, for instance by providing very helpful résumés at the ends of the chapters. Anyone who wishes to understand the relaxation methods has only (attentively, of course) to read this book, although it is doubtful whether he can appreciate their power and flexibility unless he works some examples for himself. Anyone who wishes to go further, and to use the methods for design or research, will find ample detailed instruction, including a very helpful few pages compiled from notes supplied by some members of the team which has so ably assisted Southwell in these developments. (The generous credit accorded to members of the team, both in the gracious dedication and upon every possible occasion in the text, is one graceful feature of this book.)

The 35 specific problems solved, and the diagrams in which the solutions are exhibited, form the kernel of the book. The topics cover a wide range—

elasticity, hydro- and gas-dynamics, electricity and magnetism, heat conduction, as well as formal problems solved more as a test of the methods than for the sake of the solution. The underlying theory is skilfully summarised, and the equations formulated, in each case, before the computational details are given. The diagrams record the solutions in great detail, and very clearly. The pinnacle of recorded achievement is perhaps fig. 117, a calculated " waterfall ", which might well have been used as a frontispiece to emphasise the signal power of these methods. The book well exemplifies the combination of flashes of inspiration and patient labour which are the essence of successful and sustained research. But if the labour had been mere drudgery the team could not have maintained its enthusiasm or encompassed so much.

There are two sources of error in these methods ; (i) that due to incomplete liquidation of residuals, (ii) that inherent in the finite difference approximation. The order of magnitude of (i) is apparent to the computer at every stage, and can be simply related to the accuracy desired or to the uncertainty of the data. The second can be reduced by advance to finer nets, and estimated *a posteriori*. It would, however, seem desirable (to the practical man as well as to the mathematician) that some test as to whether the error (ii) was less than (i) should be possible at any stage *without* the labour of an additional advance. Possibly a more accurate formula, using 8 instead of 4 neighbouring points, and applied only in the last stage, might serve.

Two points regarding interpolation might receive more attention. One is advice as to recommended methods of obtaining iso-curves from values computed at nodal points. The other is the calculation of values at arbitrary points within a mesh.

A minor criticism is the use by the author of the word " differential " as an abbreviation for " differential coefficient "; " derivative " would be preferable.

The Oxford University Press and the author must jointly be congratulated upon the production of a handsome volume. The Press have provided a very pleasing type-face of a size for whose return from war-time exile we are truly grateful, and paper of pre-war quality upon which to print. The author has mingled this type skilfully with italic and other founts to secure emphasis and clarity. We would, however, still recommend old face (" heads and tails ") numerals, at least for the tables in the appendix.

Finite difference approximations are not new, successive approximation is not new, Hardy Cross had an idea closely akin to that of Southwell, and many have obtained numerical solutions on square lattices, so that the novelty of the " relaxation " method has sometimes been questioned. But who else has produced solutions of such a diverse variety of problems as this book displays? We now await the promised third volume, confident that the high standard so far set can and will be maintained, and that the relaxation methods will retain their power and fertility when confronted by the greater difficulty and complexity of biharmonic and similar analysis. W. G. B.

The methods of plane projective geometry based on the use of general homogeneous coordinates. By E. A. MAXWELL. Pp. 230, xx. 12s. 6d. (Cambridge)

This is a book for pupils in their last year at school or their first year at the university. It is true to its title, being genuinely a book about methods and not merely about properties of conics, though these are sufficiently treated by way of illustrations of the methods. It is clearly written and ought to be useful to many students. Any difficulties that they may find will be due to lack of background rather than to intrinsic difficulties of the text. To say that " the reader should become familiar at the earliest possible moment with

the idea of duality " is an understatement : he ought really to have been becoming familiar with it for the past year or two ; because the acquisition of a properly constituted dual mind is not a process that can be rushed. For this and similar reasons, the reviewer feels that there are topics in this book which ought to be taken up at an earlier stage than the last year at school. On the other hand, if a student has been brought up with a hopelessly biased metrical outlook, the book will be an excellent corrective to be applied before the university stage.

There are chapters on homogeneous coordinates applied to the line, on one-one algebraic correspondence, and on cross-ratio and harmonic ranges. The conic is then treated parametrically and also from the locus and the envelope point of view, including the special equations referred to convenient triangles of reference. Then follow chapters on the quadrangle and quadrilateral, pencils of conics, and properties such as the harmonic locus and envelope, and out-polar and in-polar conics. The spirit of the book is well illustrated by the titles of the *last* two chapters which are " Relation to Euclidean Geometry " and " Applications to Euclidean Geometry " ; and it is illustrated too by the fact that there are no diagrams. The author recognises, however, that the geometer should move freely in both pure and analytical geometry, and he has not hesitated to use methods of pure geometry where they seemed most suitable to his purpose. But in fact they do not occur much. No doubt the student who solved the examples would find some occasion for them. There are some 400 examples arranged in the chapters and intended in the first place to be solved by the methods that have been introduced, but available for investigation by miscellaneous methods when the student is sufficiently advanced. There are illustrative examples in the text, and these are often standard theorems. These are generally worked out, no doubt intentionally, as straightforward applications of the methods in hand, and there is no attempt at finesse. For example, in verifying Desargues' Theorem for the triangle of reference XYZ and the triangle $X'Y'Z'$ with vertices $(\alpha+\lambda, \beta, \gamma)$, $(\alpha, \beta+\mu, \gamma)$, $(\alpha, \beta, \gamma+\nu)$, the equation of $Y'Z'$ is found by a determinant and so its point of intersection with YZ is found to be $(0, \mu, -\nu)$ which lies on $x/\lambda+y/\mu+z/\nu=0$. It is aesthetically more satisfactory to say that $(0, \mu, -\nu)$ lies on $Y'Z'$ by the definition of a line which is given on page 4 ; it also visibly lies on YZ and on $x/\lambda+y/\mu+z/\nu=0$. The proof is then essentially the same as a proof by Baker's symbols : and it is a piece of technique that is often useful with homogeneous coordinates.

The student in his last year at school may be supposed to know already something about infinity in the algebraic sense ; but if he does not, the brief outline in the introduction will not greatly help him. Indeed the statement that θ is infinite means that θ can be expressed in the form θ_1/θ_2 where $\theta_2=0$ comes dangerously near to saying that it means nothing at all, unless θ_1/θ_2 denotes a number-pair, which it does not seem to do in the context.

Chapter VII includes a proof of Chasles' Theorem $P(ABCD)=k$, followed by its converse where A, B, C, D are fixed and the locus of P is a conic through them. There is much to be said for starting with the locus of the point of intersection of corresponding lines of homographic pencils, and to relegate $P(ABCD)=k$ to the role of converse. This seems more in line with the dynamic aspect of homography between variables P, P' rather than equality of cross-ratios of fixed elements.

These small criticisms are probably little more than a reflexion of differences of opinion between one teacher and another. The fact remains that the book is likely to be useful to a large class of students, not excluding those who, in some small schools, have to manage with the minimum of mathematical teaching.

A. R.

Vorlesungen über Differential- und Integralrechnung. I. Von A. OSTROWSKI. Pp. xii, 373. Sw. fr. 47.50. 1945. (Birkhäuser, Basel)

There are many good textbooks on the calculus in English, French and German, but even on so well-worn a topic we expect Professor Ostrowski's wide and profound knowledge and technical ingenuity to provide fresh interest. This he does, in a clear, brightly-written, skilfully arranged volume. There is no intention of supplying a portentous and exhaustive treatise; the book is based on lectures and has the characteristics of such a foundation. The aim is to provide a bridge between intuitive percepts and mathematical concepts, and the best sections in the book are those which synthesise the ϵ-definitions of continuity, limit, definite integral, from the "rough idea".

The chapter topics are: 1. Introduction; 2. Limits; 3. Continuous functions and the definite integral; 4. The derivative and fundamental theorems of the calculus; 5. The technique of differentiation; 6. The technique of integration; 7. First applications of the differential calculus.

Noteworthy major points are:

(i) The use of null sequences preparatory to the general sequence limit. English texts have made some little use of this arrangement, though several have fallen into the trap of overlooking the need to *prove* results about null sequences.

(ii) The synthesis and partial analysis of the idea of the definite integral following on the heels of continuity, before discussion or definition of the derivative. Both historical and logical reasons can be advanced for this arrangement. But it is difficult to assess its true worth, since it is unlikely that either concept will be novel to the reader at this level. We may also notice the reasonable distinction drawn between concept and technique.

(iii) The postponement to a promised second volume of proofs of certain difficult but fundamental theorems, such as the general principle of convergence and Heine's theorem on the uniformity of continuity. This, however, can be overdone; surely when Heine's result is granted, proofs of the classical theorems that a continuous function has bounds which it attains, that a continuous function can not change sign without passing through zero, and so on, are all simple enough to be included in this volume, where they would be valuable in encouraging the novice to analyse an "obvious" truth.

Some smaller points may be noticed. The introduction contains a few short but most interesting paragraphs on the meaning and method of mathematics. The theory of the real number (complex numbers are barely mentioned) is shown to be necessary and a set of working rules is given and exemplified; for proof we are referred to Landau's *Grundlagen der Analysis*. The section on inequalities is brief, elementary but excellent; most of our teachers will heartily agree with Harold and Bertha Jeffreys, *Methods of Mathematical Physics*, when they say (p. 8): "Most beginners find it much more difficult to handle inequalities correctly than equalities." It is, I believe, true that many students find epsilon-techniques difficult, not because of the ideas, but because these must be presented by means of a machinery of modulus signs and inequalities which is unfamiliar and therefore distracting. The section here provided should be ample to get rid of this feeling. On the debit side, the novice whose handling of partial fractions is clumsy will not mend his ways from anything he reads here.

Two final comments. The examples, adequate but not overdone, are delightfully fresh, instructive and interesting. The typography and production are luxurious, and the book-lover will rejoice over a volume which shows no signs of "austerity".

The book can be warmly recommended as reading for the teacher or for the student just about to go up to the university. T. A. A. B.

College Algebra. By A. ADRIAN ALBERT. Pp. xii, 278. 14s. 1946. (McGraw-Hill)

An elementary textbook by a distinguished authority is always of great interest. The wide background of the expert provides him with a grasp of his subject and a feeling for its principles which the mere writer of textbooks, however much more facile he may be, will probably lack. The study of algebra in this country has for many years been comparatively neglected, and the understanding even of elementary algebraic ideas is not widespread, so that Professor Albert's book is very timely. Although it is designed as a course-text for use under American conditions (it is perhaps pitched a little above the level of the intermediate year in our universities), it is certainly relevant to the problems of teachers here.

According to the author's preface, the book is the outcome of fifteen years, reflection on the " possibility of reorganising the material of college algebra so as to present it as a sound and unified whole ". There can be few teachers who would not profit by a careful study of this book, though it is not to be expected that many would wish to adopt it in its present form as a textbook for their classes. The interest of the book lies chiefly in the exposition, although the examples are much less feeble than is usual in American works. However, the emphasis on theory rather than examples—so long as it does not become a general practice—is not in itself a matter for criticism : it is certainly arguable that the only weakness of Hardy's *Pure Mathematics* is that the magnificent collection of examples he gives serves only too often to divert the student's attention from the desirability of mastering the text.

The limited aim of the work has not been allowed to confine the subject-matter too closely : every chapter contains optional sections not required for the basic course. The exposition is, on the whole, perfectly clear. There is no possibility of mistake as to which things the author is attempting to prove and which he assumes, but it is unfortunate that these latter are never supported by any specific references to other works. For example, the method used to introduce real numbers is an attractive one, but it is not carried through in detail, and one would have wished to know where to find this done. The lay-out of the worked examples is noteworthy and wholly admirable, and the whole book has a pleasing appearance, but it is regrettable that the accuracy of the printing is so much below the standard we have come to expect from these admirable publishers.

The book divides fairly naturally into three parts. Chapters, I, II and III on number systems, IV, VI and VII on polynomials and theory of equations, and VIII, IX and X on vectors, matrices and determinants, and sets of linear simultaneous equations. The remaining Chapter (V) on " Identities and Applications " is the least satisfactory part of the book, though it is not easy to say whether this is the cause or the result of its comparative isolation. The exclusion of partial fractions from this chapter is a little difficult to understand. It may be that the author felt that this subject needed to follow the work on simultaneous equations, but the very casual treatment it receives at the end of Chapter IX arouses the suspicion that—for all his fifteen years' contemplation—the subject had slipped his mind till a very late stage in the printing. There is, however, little else to criticise. Chapter VI, on equations, might well have been somewhat expanded, and, in particular, more play could have been made with the symmetric functions of the roots ; and in the same chapter some mystery is made of the very useful process of " synthetic division " (technically and linguistically an improvement on " detached co-efficients "), which some students may perhaps fail to recognise as a shorthand method of carrying out the division of a polynomial by a linear factor.

Nevertheless the defects of this book must not be allowed to overshadow its merits. It begins briskly, the first chapter on natural numbers winding up with a treatment of permutations and combinations. But the chief value of the first part is in the insight it provides into the fundamental processes of arithmetic. A lack of appreciation of the rules for addition and multiplication is often a serious handicap in more advanced work, hindering, for instance, a grasp of operational technique and an understanding of three-dimensional vector analysis (in which a fundamental role is played by the distributive laws for vector and scalar products). The treatment of factors and of H.C.F. and L.C.M. processes is clear and satisfying and this is later effectively echoed in the chapter on polynomials. Outstanding in the second part is Chapter VII on the real roots of real polynomials, which includes Descartes' rule of signs and a statement of Sturm's theorem, as well as an account of the more common methods of approximating to irrational roots. The fundamental theorem of algebra is assumed, but the theory is deduced from this by algebraic techniques. From the strictly logical point of view this may be open to objection. The proof, for instance, of the theorem that a polynomial has a root between any two values at which it takes opposite signs, is only apparently simpler than the result for arbitrary continuous functions, since the fundamental theorem is only demonstrable by comparatively advanced methods. Nevertheless something is gained by the procedure adopted—an appreciation of algebraic methods and of the properties which distinguish polynomials from other continuous functions.

The third section is the most ambitious part of the book. In the author's own words "Chapter VIII on Vectors in the Plane is intended to be treated as optional material. It is hoped that this chapter will provide a basis for a future unification of college mathematics. The chapter begins with a derivation of the parallelogram law for the addition of vectors and the connection of unit vectors with trigonometric functions. The parallelogram law is then used in a derivation of the formulas for the rotation of axes in plane analytic geometry. These formulas yield the addition laws of trigonometry as well as de Moivre's theorem for complex numbers .... The connection of these topics shows how closely knit college algebra, trigonometry and analytic geometry really are ...." The pace is maintained in the next chapter which deals with linear systems of equations and determinants, together with the idea of a matrix considered as an array of coefficients. The final chapter, all optional, is an introduction to the technique of matrices and quadratic forms.

D. B. S.

Differential and Integral Calculus. By Ross R. Middlemiss. Pp. viii, 497. 16s. 6d. 1946. (McGraw Hill, New York and London)

Intermediate Algebra. By Neil McArthur and Alexander Keith. Pp. x, 449. 8s. 6d. (Methuen)

These books have in common the property of being second editions of works which, when first issued, were commended in this journal. The favourable comments made earlier * on the American volume can be repeated. It contains little that is novel by way of treatment, but the contents are well chosen, the appearance of text and diagrams is admirably clear, and the examples are abundant and accurate. A chapter on the elements of three-dimensional analytical geometry has been added, and provides a useful introduction to the sections on partial differentiation and multiple integration.

More changes are apparent in the book on Algebra. Chapters have been incorporated on convergency of series, exponential and logarithmic series,

* Gazette, XXV, 187, July 1941.

higher inequalities, theory of equations and determinants. To make room for these, much of the more elementary work appears in condensed form, incidentally so meeting a minor criticism made of the first edition.* Likewise we now find that "the progressions are taken for granted". Other alterations involve the section on complex numbers, which is more complete than before, and the chapters on the theory of polynomials where the order of the items has been largely rearranged.

Once more, attention should be drawn to the excellence of the presentation. The preface suggests that the text "covers, in a single volume, every form of Higher School Certificate, though it does not profess to cater completely for Scholarship candidates. The additions should also render the book more useful to university students, whose needs have been kept in mind throughout". Finally, it may be noted that the price of this second edition—which can be thoroughly recommended—is the same as that of the original volume, although the size is increased by nearly 100 pages. F. W. K.

Logarithms made easy. By H. WEBB. Pp. 31. 1s. 3d. 1946. (University of London Press)

This little book explains very clearly how to use logarithms and obtains the necessary formulae by generalising the results for positive integral exponents. It is thus suited to those who desire to obtain a good working knowledge of logarithms without entering into the more theoretical aspects of the subject.

The print is clear; four-figure logarithm and antilogarithm tables and answers to the exercises are provided. L. M. H.

What's your opinion? By W. A. BAGLEY. Pp. 107. 2s. 1946. (Vawser & Wiles)

Books presenting odd bits of information are popular and often prove of value as stimulants. This publication contains the usual collection of catch questions, jokes and out-of-the-way facts. The author disclaims omniscience, but his statements of fact are usually correct, though his explanations are not always trustworthy. In the section on "Calculations", we find some old friends, such as the donkey tethered to the edge of a circular field, with the curious comment that this is a problem "where Senior Wranglers fail". Constructive criticism is requested; we suggest that the book would be improved by omitting the travesty of spherical trigonometry given on p. 73. T. A. A. B.

Studies in science. Edited by W. C. COKER. Pp. viii, 375. 18s. 6d. 1946. (University of North Carolina Press; Oxford University Press)

Vol. LXI of the *Journal of the Elisha Mitchell Scientific Society* now appears as a special volume in the series of Sesquicentennial Publications of the University of North Carolina. The 31 contributors are all connected with the University, and their studies range over many scientific fields. Five papers deal with mathematical topics. E. T. Browne, "Concerning a certain ring of homographies", discusses a problem in matrix algebra with applications to algebraic geometry. Archibald Henderson, "The geometry of tensors of the first order", follows an idea of Appell's to give a geometric and diagrammatic study of tensor concepts. J. W. Lasley, "On the equations of certain osculants", deals with osculating curves and aberrancy of curvature, a topic rather out of fashion at present. N. Rosen, "On waves and particles", is a short note on the place of classical dynamical concepts in quantum mechanics.

* Gazette, XXVI, 155, July 1942.

A. Brauer, " A problem of additive number theory ", deals with a generalisation of a constructional problem in electrical engineering. A variable resistance was to be made, the whole resistance to be 30 ohms, and a number of fixed contact points were required, so that each integral resistance, 1, 2, 3, ... 30 ohms, could be obtained by a junction between two contacts. The smallest possible number of contact points is required.

The volume is beautifully produced and clearly printed. T. A. A. B.

Physikalische Aufgabensammlung. VON A. LÄUCHLI und F. MÜLLER. Pp. 194. Fr. Sw. 5.80. 1946. (Orell Füssli Verlag, Zürich)

This book of examples in Physics for the " Mittelschulen " of Switzerland was produced to replace standard foreign books rendered unobtainable by war-time conditions. The Swiss student must regard its appearance as one of the happier results of the war ; for in it are assembled one thousand excellent examples in nine branches of Physics : Statics, Dynamics, Elasticity, Hydrostatics, Hydrodynamics, Heat, Wave Theory and Sound, Light, Electricity. Numerical answers to the problems are given at the end of the book, and where the solution is in any way complicated the final formula is added.

The problems are set so as to stimulate thought about the processes involved rather than a search for the right formula.

A useful set of tables of physical constants and relationships is included.

The book would prove invaluable to German-reading students of intermediate and second-year standard. G. F. N.

1525. Sydney Smith in a letter to a child : And Lucy, dear child, mind your arithmetic. You know, in the first sum of yours I ever saw, there was a mistake. You had carried two (as a cab is licensed to do) and you ought, dear Lucy, to have carried but one. Is this a trifle ? What would life be without arithmetic but a scene of horrors ?—Hesketh Pearson, *The Smith of Smiths* (Hamilton, 1945), p. 225. [Per Mr. S. Melmore.]

1526. Fortunately I learned a good many years ago that mathematicians are not always the strictly accurate creatures of popular belief. Being in Paris, I developed a cold, and fearing a high temperature, bought a thermometer, which was naturally marked on the Centigrade scale, assuming that my companion, who had taken a first class in the Mathematical Tripos, would be able to translate it into Fahrenheit for me. When I asked him to do so he looked a little dubious, but sat down with pencil and paper. At the end of half an hour, he presented for my choice a variety of results, ranging from 93 deg. to 108 deg. F.—Edward Shanks, in *Sunday Times*, February 10, 1946 ; a Review of H. McKay, *The World of Numbers*.

1527. I have found out what a " difficult " proof is ; it is one which you didn't learn yourself when at school or have not taught for ten years. That is the only definition that covers all cases.—T. P. Nunn in a letter to E. H. Neville, February 25, 1924. (But I think we should add " or didn't discover for yourself ".—E. H. N.)

1528. The Ferguson Scholarships (open to graduates of the Scottish Universities), 1886. Examiner, Professor Tait. *Rubric* : Candidates are advised rather to answer some questions thoroughly, than all imperfectly. The object of the number and variety of the questions is to give (as far as possible) equal chances to students from the various Universities :—NOT to enable candidates to find occupation for three hours by selecting only the easier parts of the questions. [Per Dr. E. A. Maxwell.]

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